Processus de type Ornstein-Uhlenbeck: estimation et grandes déviations

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Processus de type Ornstein-Uhlenbeck dirigé par un brownien fractionnaire

Bernard Bercu et al. (2008) ont établi les propriétés des grandes déviations pour l'EMV et l'EB de θ dans le processus

$$dX_t = \theta X_t dt + dW_t^H, \quad \theta \leq 0, \quad 0 < H < 1$$

Généralisation

Biswal (2007) a établi un principe de grandes déviations pour l'EMV et l'EB du paramètre θ de l'EDS

$$dX_t = f(\theta, t, X_t)dt + dW_t, \quad t \geq 0, \ X_0 = 0 \tag{1}$$

Bernard Bercu et al. (2012) ont étudié le comportement asymptotique de l'EMV $\hat{\theta}_T$ du parametre reel θ

$$dX_t = \theta X_t dt + dV_t, \qquad dV_t = \rho V_t dt + dW_t \quad t \geq 0,\ X_0 = 0; V_0 = 0.$$
$$\theta < 0, \quad \rho \leq 0$$

Où $W = \{W_t,\ t \geq 0\}$ est un mouvement brownien standard.

$$dX_t = \theta X_t dt + dV_t, \qquad dV_t = \rho V_t dt + dW_t \quad t \geq 0,\ X_0 = 0; V_0 = 0\ ,$$
$$\theta < 0, \quad \rho \leq 0.$$

Bercu has prove that : $\lim_{T \to +\infty}(\hat{\theta}_T) = \theta + \rho$ a.s.
The above equation can be rewritten as :

$$dX_t = (\theta + \rho) X_t dt - \theta\rho\Sigma_t dt + dW_t, \qquad (2)$$
$$V_t = X_t - \theta\Sigma_t, \qquad (3)$$
$$\Sigma_t = \int_0^t X_s ds \qquad (4)$$

$u = (\theta + \rho; -\theta\rho)'$ and Y_t the vector $(X_t; \Sigma_t)'$, we have :

$$dX_t = u' Y_t dt + dW_t \tag{5}$$

which the general form is : $dX_t = f(u, t, X_t))dt + dW_t, \quad X_0 = 0$, where φ is a function from $\mathbb{R}$ to $\mathbb{R}^2$ which is known, W is a standard Brownian motion. The space $U = U_1 \times U_2$ where U_i is a segment of $\mathbb{R}$, the space of parameters $u = (u_1, u_2)'$ and f a function known from $U \times [0, T] \times \mathbb{R}^2$ to $\mathbb{R}$

Problemes de grandes deviations a etudier

We decide to study the large deviation of the the parameter θ for the EMV and the EB for the following EDS :

$$dX_t = f(\theta, t, X_t)dt + dW_t \tag{6}$$

$$dX_t = \theta X_t dt + dV_t, \tag{7}$$

$$dX_t = \theta X_t dt + dL_t. \tag{8}$$

Un exemple de grande deviation

Begin by studying an example. Let $(X_i)_{i\geq 1}$ a sequence of independent random variables with normal distribution centered reduced. Note $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, the mean of X_i. The weak law of large numbers ensure that : $\forall a \geq 0$, $\lim_{n\to+\infty} P(|\bar{X}_n|) \geq a) = 0$. The theory of large deviations attempts to quantify the probability of such event. $\bar{X}_n$ follows the normal law $N(0, \frac{1}{n})$ therefore $\sqrt{n}\bar{X}_n$ follows $N(0, 1)$.

Conclusion de l'exemple

We conclude that when n tends to infinity,
$P(|\bar{X}_n| \geq a) \simeq \exp(-\frac{1}{2}na^2)$.
The probability that the process $\bar{X}_n$ makes an unusual event such as exceeding his mean is approximately $\exp(-\frac{1}{2}na^2)$. With the small probability of the order of $\exp(-\frac{1}{2}na^2)$, $|\bar{X}_n|$ deviates from its typical behavior, hear his mean, by taking large values. This is the origin of the word large deviations.
The theory of the large deviations is concerned with the study of rare events and study their speed of convergence to zero in the theorems of convergence in law.

Definition de la fonction de taux

Donsker and Varadhan introduced in 1966 the statement of a principle of large deviations.

Definition (rate function)

Let X be a polish space with its Borel tribe. A fonction I from X to $[0, +\infty]$ is a function of rate if it is semi-continuous inferiorly , i.e. for all $l \geq 0$, the sets $\{x \in X : I(x) \leq l\}$ are closed in X. I is a good rate function if the sets if the levels set are compacts.

The following theorem is proved by Donsker and Varadhan in 1976.

Theorem

Let X be a separable Banach space. P a probability distribution on X which admits finite exponential moments. And let $(X_i)_{i \geq 0}$ be a sequence of independents random variables with distribution P and let be $\bar{X}_n = \frac{1}{n}\sum_{i=0}^{n} X_i$. Then for any Borel set E of X,
$-\inf\{I(x), x \in E^\circ\} \leq lim \inf \log P(\bar{X}_n \in E) \leq \lim\sup \log P(\bar{X}_n \in E) \leq -\inf\{I(x), x \in \bar{E}\}$
When I is the transform of Cramér define by :
$I(x) = \sup_{\theta \geq 0}\{\theta x - \log\{E(\exp(\theta X))\}$ *is the rate function.*

Definition generale des grandes definitions

Varhadan has given this definition of large deviations.

Definition

Let X be a polish space. We say that a family $(P^{\varepsilon})_{\varepsilon>0}$ of measures of probability satisfies a principle of large deviation with rate function I and speed of decay a_{ε} tends to zero , if :

(i) For any closed set F of X,
$\lim_{\varepsilon\to 0} \sup\ a_{\varepsilon} \log(P^{\varepsilon}(F)) \leq -\inf_{x\in F} I(x)$

(ii) For any open set of X,
$\lim_{\varepsilon\to 0} \inf\ a_{\varepsilon} \log(P^{\varepsilon}(O)) \geq -\inf_{x\in O} I(x)$

Hypotheses generales

The general form of the stochastic differential equation is : $dX_t = f(u, t, X_t)dt + dW_t, \quad X_0 = 0$, where φ is a function from $\mathbb{R}$ to $\mathbb{R}^2$ which is known, W is a standard Brownian motion. The space $U = U_1 \times U_2$ where U_i is a segment of $\mathbb{R}$, the space of parameters $u = (u_1, u_2)'$ and f a function known from $U \times [0, T] \times \mathbb{R}^2$ to $\mathbb{R}$ satisfying the Hölder continuity condition :

$$\forall u \in U; s, t \in [0, T]; (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2, \tag{9}$$

$$|f(u, t, x) - f(u, s, y)| \leq C\left(\|x - y\|^\alpha + |t - s|^\gamma\right) \tag{10}$$

$$\text{where } \|.\| \text{ is the euclidian norm.} \tag{11}$$

(A1) $P_u \neq P_v$ for $u \in U$ and $v \in U$ such as $u \neq v$.
(A2) $\{X_t\}$ is the unique strong solution of stochastic differential equation (1) with $P_u^T(\int_0^T f^2(u,t,X_t)dt < \infty) = 1$.
This condition ensures that P_u^T is absolutely continuous with respect P_W^T for all u, where P_W^T is the standard Wiener measure and likelihood function is given by :

$$L_T(u) = \frac{dP_u^T}{dP_W^T} = \exp\{\int_0^T f(u,t,X_t)dX_t - \frac{1}{2}\int_0^T f^2(u,t,X_t)dt\}, \tag{12}$$

The maximum likelihood estimator $\hat{U}_T$ of u on the basis of the observations of X_0^T defined by $\hat{U}_T = Arg\max_{u\in U} L_T(u)$.

(A3)

(i) $f(u,t,x)$ admits des continuous partial derivatives with respect to u_1 and to u_2.

(ii) Let : $I_T(u) = \int_0^T [\|Gradientf(u,t,X_t)\|]^2 dt$,
$J_w^T(w_2,z) = E_w^T \int_0^T [\frac{\partial f}{\partial u_1}(z,w_2,t,X_t)]^2 dt$ et
$J_w^T(v_1,z) = E_w^T \int_0^T [\frac{\partial f}{\partial u_2}(v_1,z,t,X_t)]^2 dt$.

(iii) The log-likelihood function has continuous partial derivatives in a neighborhood V_u of u for any u in U. Let $n_T = n_T(u) = E_u^T(I_T(u)) < \infty$ and with $n_T \to \infty$ as $T \to \infty$ and there exists a constant C_0 such that for any u, v, w belonging to U, $\frac{E_u^T(I_T(w))}{n_T(v)} < C_0$

(iv) $\frac{I_T(u)}{n_T} \xrightarrow{P_u^T} 1$ as $T \to \infty$.
(A4) Suppose there exists $\gamma \geq 2$ et $C > 0$ such that for any u $\in$ U,
$E_u^T\{\exp\left(-\frac{1}{3}\int_0^T [f(u + an_T^{-1/2}, t, X_t) - f(u, t, X_t)]^2 dt\right)\} \leq C\exp(-C(\|a\|^\gamma)$.

.

Let $p(u|X_0^T)$ be the posterior density of u given X_0^T. By Bayes theorem $p(u|X_0^T)$ is given by

$$p(u|X_0^T) = \frac{L_T(u)\lambda(u)}{\int_U L_T(u)\lambda(u)du}.$$

Let $l(.,.) : U \times U \to \mathbb{R}$ be a loss function as defined in Ibragimov and Khasminski (1981) which satisfies the following conditions :
(B1) $l(u, v) = \psi(\|u - v\|)$, where $\|.\|$ is the euclidian norm.
(B2) $\psi(u)$ is defined and nonnegative on $\mathbb{R}$, $\psi(0) = 0$ and $\psi(u)$ is continuous at $u = 0$ but is not identically equal to 0.
(B3) ψ is symmetric
(B4) $\{u : \psi(u) < c\}$ are convex sets and bounded for all $c > 0$ sufficiently small
(B5) There exists numbers $\gamma > 0$, $h_0 \geq 0$ such that for $h \geq h_0$

$$\sup\{\psi(u) : |u| \leq h^\gamma\} \leq \{\psi(u) : |u| \geq h\}.$$

L'estimateur Bayesien

A Bayes estimator $\widetilde{u}_T$ of u with respect to the loss function $l(\varphi; u)$ and prior density $\lambda(u)$ is one which minimizes the posterior risk and is given by

$$\widetilde{U}_T := arg \min_{\phi \in U} \int_U l(\phi, u) p(u|X_0^T) du.$$

In particular, for the quadratic loss function $l(u, v) = \|u - v\|^2$, the Bayes estimator $\widetilde{U}_T$ becomes the posterior mean given by :

$$\widetilde{U}_T = (\widetilde{\alpha}_T^1, \widetilde{\alpha}_T^2), \widetilde{\alpha}_T^i = \frac{\int_U u_i p(u|X_0^T) du}{\int_U p(u|X_0^T)}$$

Theorem

With $dX_t = u'X_t dt + dW_t,$ *the maximum likelihood is :* (13)

$$L_T(u) = \exp(\int_0^T (u'X_t)dX_t - \frac{1}{2}\int_0^T (u'X_t)^2 dt) \tag{14}$$

The maximum likelihood estimator of u is :

$$\hat{U_T} = (\int_0^T X_t X_t' dt)^{-1} \int_0^T X_t dX_t \tag{15}$$

Theorem

the maximum likelihood estimator of u is strongly consistent.

Bernard Bercu, Frédéric Proia and Nicolas Savy established that :

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T X_t dX_t = -\frac{1}{2} \quad a.s., \quad (16)$$

$$\lim_{T\to+\infty} \frac{P_T}{T} = -\frac{1}{2(\theta+\rho)} \quad a.s., \quad (17)$$

$$\lim_{T\to+\infty} \frac{S_T}{T} = -\frac{1}{2(\theta+\rho)} \quad a.s., \quad (18)$$

$$\lim_{T\to+\infty} \frac{L_T}{T} = -\frac{1}{2\rho} \quad p.s., \quad (19)$$

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T V_t dV_t = -\frac{1}{2} \quad a.s., \quad (20)$$

$$\int_0^T V_t dV_t = \rho L_T + M_T^V, \quad \int_0^T X_t dX_t = \theta S_T + \rho P_T + M_T^X \quad (21)$$

Lemma (Ibragimov and Khasminskii (1981, p.45))

Let $Z_{\varepsilon,u}(y)$ be the likelihood ratio function corresponding to the points $u+\phi(\varepsilon)y$ and u where $\phi(\varepsilon)$ denotes a normalizing factor such that $|\phi(\varepsilon)| \to 0$ as $\varepsilon \to 0$. Thus $Z_{\varepsilon,u}$ is defined on the set $U_\varepsilon = (\phi(\varepsilon))^{-1}(U-u)$. Let $Z^u_{\varepsilon,u}(t)$ possesses the following properties : given a compact set $K \subset U$ there exist numbers $M_1 > 0$ and $m_1 \geq 0$ and functions $g^K_\varepsilon(y) = g_\varepsilon(y)$ correspond such that :

(1) *For some $\alpha > 0$ and all $u \in K$,*

$$\sup_{\substack{\|a\|\leq R\\ \|b\|\leq R}} \|a-b\|^{-\alpha} E_u^{(\varepsilon)} \left| Z^{\frac{1}{2}}_{\varepsilon,u}(b) - Z^{\frac{1}{2}}_{\varepsilon,u}(a) \right|^2 \leq M_1(1+R^{m_1})$$

(2) *For all $u \in K$ and $y \in U_\varepsilon$, $E_u^{(\varepsilon)} Z^{\frac{1}{2}}_{\varepsilon,u}(y) \leq e^{-g_\varepsilon(y)}$*

Lemma

$$\lim_{\substack{y \to \infty \\ \varepsilon \to 0}} y^N e^{-g_\varepsilon(y)} = 0$$

Let $\{\widetilde{u}_\varepsilon\}$ be a family of Bayes estimators with respect to the prior density q, which is continuous and positive on K and possesses in U a polynomial majorant and a loss function $w_\varepsilon(u, v) := \psi\left((\phi(\varepsilon))^{-1}(u - v)\right)$ where ψ satisfies $(B1) - (B5)$. Then for all N,

$$\lim_{\substack{y \to \infty \\ \varepsilon \to 0}} h^N \sup_{u \in K} P_u^{(\varepsilon)} \left\{ \left\| (\phi(\varepsilon))^{-1} (\widetilde{u}_\varepsilon - u) \right\| > h \right\} = 0.$$

If in addition, $\psi(y) = \tau(|y|)$, then for all ε sufficiently small, $0 < \varepsilon < \varepsilon_0$,

$$\sup P_u^{(\varepsilon)} \left\{ \left\| (\phi(\varepsilon))^{-1} (\widetilde{u}_\varepsilon - u) \right\| > h \right\} \le B_0 e^{-b_0 g_\varepsilon(h)}.$$

Lemma

Under the assumptions (A1)-(A4), and posing : $u = (u_1, u_2)'$, $v = (v_1, v_2)'$, $w = (w_1, w_2)'$, $\delta_t = f(w, t, Y_t) - f(v, t, Y_t)$.

$V_T = \left(\frac{dP_w^T}{dP_v^T}\right)^{1/2}$.

$$E_w^T\left(\int_0^T \delta_t^2 dt\right) \leq 2(w_1 - v_1)\int_{v_1}^{w_1} J_w^T(w_2, z)dz + 2(w_2 - v_2)\int_{v_2}^{w_2} J_w^T(v_1, z)dz \qquad (22)$$

Démonstration.

$V_T = \exp\{\frac{1}{2}\int_0^T \delta_t dW_t - \frac{1}{4}\int_0^T \delta_t^2 dt\}$.
Using Itô's formula, we have :
$V_T = 1 + \frac{1}{2}\int_0^T V_t\delta_t dW_t - \frac{1}{8}\int_0^T V_t\delta_t^2 dt$.
The stochastic process $\{V_t^2, \mathcal{F}_t, P_u^T; 0 \leq t \leq T\}$ is a martingale and by the $\mathcal{F}_t$ measurability of δ_t for any t belonging to the interval $[0, T]$, we have :

$$E_v^T[V_T^2\delta_t^2|\mathcal{F}_t] = E_v^T[V_T^2|\mathcal{F}_t]\delta_t^2$$

.
We have : □

Démonstration.

$$\begin{aligned}
E_v^T \int_0^T V_t^2 \delta_t^2 dt &= \int_0^T E_v^T (V_t^2 \delta_t^2) dt \\
&= \int_0^T E_v^T (E_v^T [V_T^2 | \mathcal{F}_t] \delta_t^2) dt \\
&= \int_0^T E_v^T (E_v^T [V_T^2 \delta_t^2 | \mathcal{F}_t]) dt \\
&= \int_0^T E_v^T (V_T^2 \delta_t^2) dt
\end{aligned}$$

□

Démonstration.

$$= E_v^T \left(\int_0^T V_T^2 \delta_t^2 dt \right)$$

$$= E_v^T V_T^2 \left(\int_0^T \delta_t^2 dt \right)$$

$$= \int V_T^2 \left(\int_0^T \delta_t^2 dt \right) dP_v^T$$

$$= \int \left(\int_0^T \delta_t^2 dt \right) dP_w^T, \qquad \text{by definition of } V_t$$

$$= E_w^T \left(\int_0^T \delta_t^2 dt \right)$$

□

Démonstration.

$$
\begin{aligned}
&= E_w^T\left(\int_0^T [f(w,t,X_t) - f(v,t,X_t)]^2 dt\right) < \infty \\
&= E_w^T\left(\int_0^T [f(w_1,w_2,t,X_t) - f(v_1,v_2,t,X_t)]^2 dt\right) \\
&= E_w^T \int_0^T \{\int_{v_1}^{w_1} \frac{\partial f}{\partial u_1}(z,w_2,t,X_t)dz + \int_{v_2}^{w_2} \frac{\partial f}{\partial u_2}(v_1,z,t,X_t)dz\}^2 dt. \\
&\leq 2E_w^T \int_0^T [\int_{v_1}^{w_1} \frac{\partial f}{\partial u_1}(z,w_2,t,X_t)dz]^2 + [\int_{v_2}^{w_2} \frac{\partial f}{\partial u_2}(v_1,z,t,X_t)dz]^2 dt.
\end{aligned}
$$

□

Démonstration.

$$\leq 2(w_1 - v_1)E_w^T \int_0^T \int_{v_1}^{w_1} (\frac{\partial f}{\partial u_1}(z, w_2, t, Y_t))^2 dzdt +$$

$$2(w_2 - v_2)E_w^T \int_0^T \int_{v_2}^{w_2} (\frac{\partial f}{\partial u_2}(v_1, z, t, Y_t))^2 dzdt.$$

$$\leq 2(w_1 - v_1)E_w^T \int_{v_1}^{w_1} \int_0^T (\frac{\partial f}{\partial u_1}(z, w_2, t, Y_t))^2 dtdz +$$

$$2(w_2 - v_2)E_w^T \int_{v_2}^{w_2} \int_0^T (\frac{\partial f}{\partial u_2}(v_1, z, t, Y_t))^2 dtdz.$$

□

Démonstration.

$$\leq 2(w_1 - v_1) \int_{v_1}^{w_1} \int_0^T E_w^T (\frac{\partial f}{\partial u_1}(z, w_2, t, Y_t))^2 dtdz +$$

$$2(w_2 - v_2) \int_{v_2}^{w_2} \int_0^T E_w^T (\frac{\partial f}{\partial u_2}(v_1, z, t, Y_t))^2 dtdz.$$

$$\leq 2(w_1 - v_1) \int_{v_1}^{w_1} J_w^T(w_2, z)dz + 2(w_2 - v_2) \int_{v_2}^{w_2} J_w^T(v_1, z)dz < \infty.$$

$$E_w^T (\int_0^T \delta_t^2 dt) \leq 2(w_1 - v_1) \int_{v_1}^{w_1} J_w^T(w_2, z)dz +$$

$$2(w_2 - v_2) \int_{v_2}^{w_2} J_w^T(v_1, z)dz$$

□

Definition de la fonction rapport de vraisemblance

$$\Psi_T(a) = \frac{dP^T_{u+an_T^{-1/2}}}{dP^T_u}(X_0^T). \tag{23}$$

We have :

$$\Psi_T(a) = \exp[\int_0^T g_t(a)dW_t - \frac{1}{2}\int_0^T g_t^2(a)dt]. \tag{24}$$

Lemma

Under the assumptions (A1)-(A4),

$$a)\ \sup_{u\in U} E_u^T\left[\Psi_T^{1/2}(a)-\Psi_T^{1/2}(b)\right]^2 \leq \frac{C_0}{2}\|a-b\|^2; \tag{25}$$

$$b)\ \sup_{u\in U} E_u^T\left[\Psi_T^{1/2}(a)\right] \leq C\exp(-C\|a\|^{\gamma}); \tag{26}$$

$$c)\ \sup_{u\in U} E_u^T\left[\Psi_T^{1/8}(a)-\Psi_T^{1/8}(b)\right]^4 \leq 2K\|a-b\|^4; \tag{27}$$

Démonstration.

$$u = (u_1, u_2)', v = (v_1, v_2)', w = (w_1, w_2)', a = (a_1, a_2)', b = (b_1, b_2)',$$

$$v = u + a n_T^{-1/2}, w = u + b n_T^{-1/2}, \delta_t = f(w, t, Y_t) - f(v, t, Y_t).$$

□

Démonstration.

$$E_u^T[\Psi_T^{1/2}(a)\Psi_T^{1/2}(b)] = E_u^T\left[\frac{dP^T_{u+an_T^{-1/2}}}{dP_u^T}\right]^{\frac{1}{2}}\left[\frac{dP^T_{u+bn_T^{-1/2}}}{dP_u^T}\right]^{\frac{1}{2}}$$

$$= \int\left[\frac{dP_v^T}{dP_u^T}\right]^{\frac{1}{2}}\left[\frac{dP_w^T}{dP_u^T}\right]^{\frac{1}{2}} dP_u^T$$

$$= \int\left[\frac{dP_w^T}{dP_v^T}\right]^{\frac{1}{2}} dP_v^T = E_v^T(V_T) \qquad (28)$$

□

Démonstration.

$$E_u^T(\Psi_T^{1/2}(a) - \Psi_T^{1/2}(b))^2 \leq 2 - E_v^T(V_T)$$

$$\leq \frac{1}{8} E_v^T \int_0^T \delta_t^2 dt + \frac{1}{8} E_w^T \int_0^T \delta_t^2 dt$$

$$E_v^T(\int_0^T \delta_t^2 dt) \leq 2(w_1 - v_1) \int_{v_1}^{w_1} J_v^T(v_2, z) dz$$

$$+2(w_2 - v_2) \int_{v_2}^{w_2} J_v^T(w_1, z) dz$$

$$E_u^T(\Psi_T^{1/2}(a) - \Psi_T^{1/2}(b))^2 \leq \frac{1}{4}(w_1 - v_1) \int_{v_1}^{w_1} [J_w^T(w_2, z) + J_v^T(v_2, z)] dz$$

$$+\frac{1}{4}(w_2 - v_2) \int_{v_2}^{w_2} [J_w^T(v_1, z) + J_v^T(w_1, z)] dz$$

Démonstration.

$$\begin{aligned}
&\leq \frac{1}{2}(w_1 - v_1)^2 \sup_{u,z} J_u^T(u,z) + \frac{1}{2}(w_2 - v_2)^2 \sup_{u,z} J_u^T(u,z) \\
&\leq \frac{1}{2}[(w_1 - v_1)^2 + \frac{1}{2}(w_2 - v_2)^2] \sup_{u,z} J_u^T(u,z) \\
&\leq \frac{1}{2}\|w - v\|^2 \sup_{u,z} J_u^T(u,z) \\
&\leq \frac{1}{2}\frac{\|a - b\|^2}{n_T} \sup_{u,z} J_u^T(u,z) \\
E_u^T(\Psi_T^{1/2}(a) - \Psi_T^{1/2}(b))^2 &\leq \frac{C_0}{2}\|a - b\|^2.
\end{aligned}$$

□

Principe des grandes deviations pour l'estimateur Bayesien

Theorem

Suppose (A1)-(A6) and (B1)-(B5) hold. For $\rho > 0$, the Bayes estimator $\widetilde{\theta}_T$ with respect to the prior $\lambda(.)$ and a loss function $l(.,.)$.

$$\sup_{u \in U} \mathbb{P}_u^T \left\{ \sqrt{T} \| \widetilde{U}_T - u \| > \rho \right\} \leq B \exp(-b\rho^\gamma)$$

for some positive constant B and b dependent of ρ and T.

Démonstration.

Using Lemma of Ibragimov), we obtain the lemma 3 (1) with $\alpha = 2$, $Z_{\varepsilon,u} = \Psi_T$, $M_1 = \frac{C_0}{2}$ and $m_1 = 0$. Using Lemma 5 b), we obtain the lemma 3 (2) with $Z_{\varepsilon,u} = \Psi_T$ and $g_\varepsilon(y) = y^\gamma$. Using Lemma 5 b), we obtain the lemma 3 (3) with $g_\varepsilon(y) = y^\gamma$.
Then with $\phi(\varepsilon) = \frac{1}{\sqrt{T}}$, and taking into all the above, we have the required result. □

THANK YOU