Couplage du principe des grandes déviations et de l'homogénisation dans le cas des EDP paraboliques: (le cas constant)

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Plan

History

History

- P. Baldi (1991)
- Freidlin and Sowers (1999)
- Diédhiou and Manga (2007)
- Diédhiou (Cimpa 2014)

Introduction

abstract

We consider the coupling of homogenization and large deviation principle in partial differential equation (PDE). We compare them with the help of the ratio δ/ϵ between the small viscosity parameter (ϵ) and the homogenization parameter (δ); this comparison is required as ϵ and δ tend to zero. We use some large deviation estimates to study the behavior of the PDE solution.

PDE

PDE

$$\begin{cases} \dfrac{u^{\epsilon,\delta}}{\partial t}(t,x) = L_{\epsilon,\delta} u^{\epsilon,\delta}(t,x) + \frac{1}{\epsilon} f\left(\frac{x}{\delta}, u^{\epsilon,\delta}(t,x)\right) \\ u^{\epsilon,\delta}(0,x) = g(x), \qquad x \in \mathbb{R}^d \end{cases} \tag{1}$$

assumption

f is a nonlinear function 1-periodic and verify:

- for all x, $f(x, 1) = 0$
- there is c a bounded function : $f(x, y) = c(x, y).y$

and

- $c(x, y) > 0$ for all x and y in (0,1)
- $c(x, y) \leq 0$ if not
- $\max c(x, y) = c(x)$

and
$g \in C\left(\mathbb{R}^d, \mathbb{R}^+\right)$ a bounded function : $\sup_{x \in \mathbb{R}^d} g(x) = \bar{g} < \infty$.
Take $G_0 = \left\{x \in \mathbb{R}^d : g(x) > 0\right\}$, g is continuous one notes

$$\overline{\mathring{G}_0} = \overline{G_0}$$

definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability triple on which a d-dimensional Brownian motion $\left(W^1, \ldots, W^d\right)$ is defined. Let $\mathbb{E}$ the corresponding expectation operator. We have already defined $\langle ., . \rangle$ as the standard euclidian inner product on $\mathbb{R}^d$; let $\| \, . \, \|$ be the associated norm. Also let $\mathbb{T}^d$ be the d-dimensional torus of size 1 and $C\left(\mathbb{T}^d; \mathbb{R}^d\right)$ be the space of continuous mapping from $\mathbb{T}^d$ to $\mathbb{R}^d$; let $\| \, . \, \|_{C\left(\mathbb{T}^d;\mathbb{R}^d\right)}$ be the associated supremum norm. Also we define $\mathcal{P}\left(\mathbb{T}^d\right)$ as the collection of all probability measure on $\mathbb{T}^d$.

Approach

Consider the Markov diffusion process $X_t^{x,\epsilon,\delta} \in \mathbb{R}^d$ governed by the operator :

$$\boldsymbol{L}_{\epsilon,\delta} = \frac{\epsilon}{2}\sum_{i,j=1}^{d}(\sigma\sigma^*)_{ij}\left(\frac{x}{\delta}\right)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} B^{\epsilon,\delta}\left(\frac{x}{\delta}\right)\frac{\partial}{\partial x_i}$$

The trajectories of this process can be constructed with the help of the SDE:

$$\begin{cases} dX_t^{x,\epsilon,\delta} = \sqrt{\epsilon}\sigma\left(\frac{X_t^{x,\epsilon,\delta}}{\delta}\right)dW_t + B^{\epsilon,\delta}\left(\frac{X_t^{x,\epsilon,\delta}}{\delta}\right)dt \\ X_0^{x,\epsilon,\delta} = x \end{cases}$$

assumption

$\sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ and $B^{\epsilon,\delta} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ are regular applications. The vector-valued function $B^{\epsilon,\delta}$ is given by :

$$B^{\epsilon,\delta} = \frac{\epsilon}{\delta} B_0 + B_1 + B_2^{\epsilon,\delta}$$

B_0, B_1 and $B_2^{\epsilon,\delta}$ are $C^\infty\left(\mathbb{R}^d, \mathbb{R}^d\right)$ for all $\epsilon, \delta > 0$ and

$$\lim_{\epsilon,\delta\to 0} \| B_2^{\epsilon,\delta} \|_{C_p\left(\mathbb{R}^d,\mathbb{R}^d\right)} = 0$$

hypothesis

Since the two parameters δ (homogenization) and ϵ (large deviation) tend to zero, we consider a new defined parameter

$$\delta_\epsilon = \delta$$

Suppose that $\lim_{\epsilon \downarrow 0} \frac{\delta_\epsilon}{\epsilon} = \gamma$, where $\gamma > 0$ a constant . There results that the homogenization parameter and the large deviation parameter go at the same rate

some rescalings

covering map

Define $\tilde{X}_t^{x,\epsilon,\delta_\epsilon}$ by : $\boldsymbol{\tilde{X}_t^{x,\epsilon,\delta_\epsilon}} = \frac{1}{\delta_\epsilon} X^{x,\epsilon,\delta_\epsilon}_{\left(\frac{\delta_\epsilon}{\sqrt{\epsilon}}\right)^2 t}$

Then

$$\begin{cases} d\tilde{X}_t^{x,\epsilon,\delta} = \sigma\left(\tilde{X}_t^{x,\epsilon,\delta}\right) d\tilde{W}_t + \frac{\delta_\epsilon}{\epsilon} B^{\epsilon,\delta}\left(\tilde{X}_t^{x,\epsilon,\delta}\right) dt \\ \tilde{X}_0^{x,\epsilon,\delta} = \frac{x}{\delta_\epsilon} \end{cases}$$

where $\tilde{W}_t^{\epsilon,\delta_\epsilon} = \frac{\sqrt{\epsilon}}{\delta_\epsilon} W^{\epsilon,\delta_\epsilon}_{\left(\frac{\delta_\epsilon}{\sqrt{\epsilon}}\right)^2 t}$ is a Brownian motion.

new consideration

the torus $\mathbb{T}^d$

Thereafter, consider the process $\left\{\tilde{X}_t^{\epsilon,\delta_\epsilon} : t \geq 0\right\}$ $\mathbb{T}^d$-valued which generator is defined by :

$$\tilde{L}_{\epsilon,\delta_\epsilon} = \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^*)_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \frac{\delta_\epsilon}{\epsilon}\sum_{i=1}^{d} B^{\epsilon,\delta_\epsilon}(x)\frac{\partial}{\partial x_i}$$

Consider $a = \sigma\sigma^*$, the above generator converges to the operator

$$L_\gamma = \frac{1}{2}\sum_{i,j=1}^{d}(a)_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} B_i^0(x)\frac{\partial}{\partial x_i} + \gamma\sum_{i=1}^{d} B_i^1(x)\frac{\partial}{\partial x_i}$$

reference

tools

The basic set of calculations of this subject involves deriving the Varadhan formula and in identifying the constant $\overline{C}_2$. The main technique for showing that is the following result : (Baxendale and Stroock (1988), corollary 1.12, p[183] to p[185])

LDP

rate function

Let $\overline{\mathcal{J}^2}(\theta)$ defined by Freidlin and Sowers (1999), the limit of

$$\epsilon \log \mathbb{E}\left[\exp\left(\frac{1}{\epsilon}\langle\theta, X^{x,\epsilon,\delta_\epsilon}\rangle\right)\right]$$

$$\overline{\mathcal{J}^2}(\theta) = \inf_{\{\phi\in C^\infty(\mathbb{T}^d)\}} \sup_{\{\mu\in\mathcal{P}(\mathbb{T}^d)\}} \int_{\mathbb{T}^d} \left(\frac{1}{2}\sum_{k=1}^{d} \langle (I-\nabla\phi)\,\sigma_k(z), \theta\rangle^2 + \langle (I-\nabla\phi)\, B_1(z), \theta\rangle \right.$$
$$\left. + \frac{1}{\gamma}\langle (B_0 - L_0\phi)(z), \theta\rangle\right)\mu(dz)$$

Define the Legendre-fenchel of $\overline{\mathcal{J}^2}(\theta)$

$$\mathcal{J}^2(\theta) = \sup_{\{\theta'\in\mathbb{R}^d\}} \left\langle \theta, \theta' \right\rangle - \overline{\mathcal{J}^2}(\theta')$$

large deviation principle

LDP

Define in addition :

$$\mathbf{S^2_{0,T}(\phi)} = \begin{cases} \int_0^T \mathcal{J}^2\left(\dot{\phi}(s)\right) ds & \text{if } \phi \text{ absolutely continuous and } \phi(0) = x \\ \infty & \text{if not} \end{cases}$$

Theorem (Freidlin and Sowers 1999)

Fix $T > 0$ and $x \in \mathbb{R}^d$. The family $\left\{X_t^{x,\epsilon,\delta_\epsilon} \; : \; 0 \le t \le T\right\}_{\epsilon>0}$ of $C\left([0,T], \mathbb{R}^d\right)$-valued random variables has a large deviation principle with rate function $S^2_{0,T}(\phi)$ for all $\phi \in C\left([0,T], \mathbb{R}^d\right)$.

barrier

constant

Now define

$$\overline{C}_2 = \sup_{\mu \in \mathcal{P}(\mathbb{T}^d)} \int_{\mathbb{T}^d} c(z)\mu(dz)$$

fundemental theorem

Varadhan formula

Theorem

Let c be an element of $C^{\infty}\left(\mathbb{T}^{d}\right)$ *and D a Borel subset of* $C\left([0,t],\mathbb{R}^{d}\right)$. *Then*

$$\liminf_{\epsilon\downarrow 0}\epsilon\log\mathbb{E}\left[\mathbf{1}_{D}\left(X_{t}^{x,\epsilon,\delta_\epsilon}\right)e^{\left\{\frac{1}{\epsilon}\int_0^t c\left(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon}\right)ds\right\}}\right]\geq t\overline{C}_2-\inf_{\left\{\phi\in\overset{\circ}{D}\right\}}S_{0,t}^{2}(\phi)$$

$$\limsup_{\epsilon\downarrow 0}\epsilon\log\mathbb{E}\left[\mathbf{1}_{D}\left(X_{t}^{x,\epsilon,\delta_\epsilon}\right)e^{\left\{\frac{1}{\epsilon}\int_0^t c\left(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon}\right)ds\right\}}\right]\leq t\overline{C}_2-\inf_{\left\{\phi\in\overline{D}\right\}}S_{0,t}^{2}(\phi)$$

behavior of $u^{\epsilon,\delta_\epsilon}$

Feynman Kac

$$\boldsymbol{u^{\epsilon,\delta_\epsilon}(t,x)} = \mathbb{E}\left[g\left(X_t^{x,\epsilon,\delta_\epsilon}\right) e^{\frac{1}{\epsilon}\int_0^t c\left(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon}, Y_s^{x,\epsilon,\delta_\epsilon}\right) ds} \right]$$

BSDE

Pardoux Peng 1992

$$\left\{\begin{array}{l} Y_s^{x,\epsilon,\delta_\epsilon} = g\left(X_t^{x,\epsilon,\delta_\epsilon}\right) + \frac{1}{\epsilon}\int_s^t f\left(\frac{X_r^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon}, Y_r^{x,\epsilon,\delta_\epsilon}\right) dr - \frac{1}{\sqrt{\epsilon}}\int_s^t Z_r^{x,\epsilon,\delta_\epsilon} dW_r \\ \mathbb{E}\left\{\int_s^t \left|Z_r^{x,\epsilon,\delta_\epsilon}\right|^2 dr\right\} < \infty \end{array}\right. , 0 \leq s \leq t$$

and

$$Y_0^{x,\epsilon,\delta_\epsilon} = u^{\epsilon,\delta_\epsilon}(t,x)$$

viscosity solution

Pradeilles 1998

Since $u^{\epsilon,\delta_\epsilon}(t,x) > 0$, denote $\boldsymbol{v^{\epsilon,\delta_\epsilon}(t,x)} = \epsilon \log u^{\epsilon,\delta_\epsilon}(t,x)$ and observe, that $v^{\epsilon,\delta_\epsilon}(t,x)$ is a viscosity solution of :

$$\begin{cases} \dfrac{\partial v^{\epsilon,\delta_\epsilon}}{\partial t}(t,x) &= L_{\epsilon,\delta_\epsilon} v^{\epsilon,\delta_\epsilon}(t,x) + \dfrac{1}{2}\left\|\nabla v^{\epsilon,\delta_\epsilon}(t,x)\sigma(x)\right\|^2 + c\left(\dfrac{x}{\delta_\epsilon}, u^{\epsilon,\delta_\epsilon}(t,x)\right) \\ v^{\epsilon,\delta_\epsilon}(0,x) &= \epsilon \log(g(x)), \qquad x \in G_0 \\ \lim\limits_{t\to 0} v^{\epsilon,\delta_\epsilon}(t,x) &= -\infty, \qquad x \in \mathbb{R}^d \backslash G_0 \end{cases}$$

some notation

definition

Let us define a distance in $\mathbb{R}^+ \times \mathbb{R}^d$, by for $(t,x),(s,y) \in \mathbb{R}^+ \times \mathbb{R}^d$:

$$\boldsymbol{d}\left\{(\boldsymbol{t},\boldsymbol{x}),(\boldsymbol{s},\boldsymbol{y})\right\} = \max\left\{|t-s|, \|x-y\|\right\}$$

Let us now introduce some notation :

$$\boldsymbol{A} = \inf_{\{\phi\in C^\infty(\mathbb{T}^d)\}} \sup_{\mu\in\mathcal{P}(\mathbb{T}^d)} \int_{\mathbb{T}^d} (I-\nabla\phi)\, a\, (I-\nabla\phi)(z)\mu(dz)$$

$$\boldsymbol{B} = \inf_{\{\phi\in C^\infty(\mathbb{T}^d)\}} \sup_{\{\mu\in\mathcal{P}(\mathbb{T}^d)\}} \int_{\mathbb{T}^d} \left[(I-\nabla\phi)\, B_1 + \frac{1}{\gamma}(B_0 - L_0\phi)\right](z)\mu(dz)$$

$$\boldsymbol{\rho^2(t,x,G_0)} = \inf_{\left\{\phi\in C\left([0,t],\mathbb{R}^d\right);\phi_0=x;\phi_t\in G_0\right\}} S_2^{0,\tau}(\phi)$$

asymptotic behavior

Define

$$\overline{\boldsymbol{v}}(\boldsymbol{t},\boldsymbol{x}) = \limsup_{\eta \to 0} \left\{ v^{\epsilon,\delta_\epsilon}(s,y) :\ (s,y) \in \mathcal{B}\{(t,x),\eta\} \right\}$$

$$\underline{\boldsymbol{v}}(\boldsymbol{t},\boldsymbol{x}) = \liminf_{\eta \to 0} \left\{ v^{\epsilon,\delta_\epsilon}(s,y) :\ (s,y) \in \mathcal{B}\{(t,x),\eta\} \right\}$$

Theorem

$\underline{v}$ and $\overline{v}$ are sub and super viscosity solutions of :

$$\begin{cases} \max_{w} \left(\frac{\partial w}{\partial t}(t,x) - \frac{1}{2} \langle A \nabla w(t,x), \nabla w(t,x) \rangle - \langle B, \nabla w(t,x) \rangle - \overline{C}_2 \right) = 0, \ x \in \mathbb{R}^d, t > 0 \\ w(0,x) \ = 0, \quad x \in G_0 \\ \lim_{t \to 0} w(t,x) \ = \ -\infty, \quad x \in \mathbb{R}^d \setminus G_0 \end{cases}$$

asymptotic behavior

definition

Let $\mathcal{O}$ is open subset in $\mathbb{R}^+ \times \mathbb{R}^d$, define the function τ on $[0\,,\infty[\,\times C\left([0\,,\infty[\,\times\mathbb{R}^d\right)$

$$\boldsymbol{\tau} = \tau(t,\phi) = \inf\{s : (t-s,\phi(s)) \in \mathcal{O}\}$$

Take $\boldsymbol{\Theta}_t$ the set of Markov functions τ.
Use the process

$$\boldsymbol{V(t,x)} = \inf_{\tau\in\Theta_t}\left\{\overline{C}_2\tau - \inf_{\{\phi\in C\left([0,t],\mathbb{R}^d\right);\phi_0=x;\phi_t\in G_0\}} S_2^{0,\tau}(\phi)\right\}$$

asymptotic behavior

using the fact (Pradeilles 1998):

$$-\rho^2\left(t,x,G_0\right) \le \underline{v}(t,x) \le \overline{v}(t,x) \le \min\left(Kt-\rho^2\left(t,x,G_0\right),0\right)$$

Theorem

For $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d$,

$$\lim_{\epsilon\downarrow 0} \epsilon \log u^{\epsilon,\delta_\epsilon}(t,x) = V(t,x)$$

asymptotic behavior

corollary

Let $\mathcal{M}$ and $\mathcal{E}$ be a partition of $\mathbb{R}^+ \times \mathbb{R}^d$, such that:

$$\mathcal{M} = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \;:\; V(t,x) = 0\right\}$$
$$\mathcal{E} = \left\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \;:\; V(t,x) < 0\right\}$$

Theorem

We have

$$\lim_{\epsilon\downarrow 0} u^{\epsilon,\delta_\epsilon(t,x)} = \begin{cases} 0 \; \textit{uniformly from any compact } \; \mathcal{K} \textit{ of } \; \mathcal{E} \\ 1 \; \textit{uniformly from any compact } \; \mathcal{K}' \textit{ of } \; \overset{\circ}{\mathcal{M}} \end{cases}$$