Couplage du principe des grandes déviations et de l'homogénisation dans le cas des EDP paraboliques:

Plan

History

History

P. Baldi (1991)

Freidlin and Sowers (1999)

Diédhiou and Manga (2007)

Diédhiou (Cimpa 2014)

Introduction

abstract

We consider the coupling of homogenization and large deviation principle in partial differential equation (PDE). We compare them with the help of the ratio = between the small viscosity parameter ( ) and the homogenization parameter ( ); this comparison is required as and tend to zero. We use some large deviation estimates to study the behavior of the PDE solution.

PDE

PDE

$$\left\{\begin{array}{ll} \dfrac{\partial u^{\epsilon,\delta}}{\partial t}(t;x) = L_{\epsilon,\delta} u^{\epsilon,\delta}(t;x) + \dfrac{1}{\epsilon} f\left(\dfrac{x}{\delta}; u^{\epsilon,\delta}(t;x)\right) & \\ u^{\epsilon,\delta}(0;x) = g(x); \qquad x \in \mathbb{R}^d & \end{array}\right. \tag{1}$$

assumption

f is a nonlinear function 1-periodic and verify:

for all x, $f(x;1) = 0$

there is c a bounded function : $f(x;y) = c(x;y){:}y$

and

$c(x;y) > 0$ for all x and y in (0,1)

$c(x;y) \leq 0$ if not

$\max c(x;y) = c(x)$

and

$g \in C\left(\mathbb{R}^d;\mathbb{R}^+\right)$ a bounded function : $\sup_{x\in\mathbb{R}^d} g(x) = \bar{g} < \infty$.

Take $G_0 = \left\{x \in \mathbb{R}^d : g(x) > 0\right\}$, g is continuous one notes

$$\overline{\mathring{G}_0} = \overline{G_0}$$

definition

Let $(\Omega; \mathcal{F}; \mathbb{P})$ a probability triple on which a d-dimensional Brownian motion $W^1; \ldots; W^d$ is defined. Let $\mathbb{E}$ the corresponding expectation operator. We have already defined $\langle \cdot; \cdot \rangle$ as the standard euclidian inner product on $\mathbb{R}^d$; let $\| \cdot \|$ be the associated norm. Also let $\mathbb{T}^d$ be the d-dimensional torus of size 1 and $C\left(\mathbb{T}^d; \mathbb{R}^d\right)$ be the space of continuous mapping from $\mathbb{T}^d$ to $\mathbb{R}^d$; let $\| \cdot \|_{C(\mathbb{T}^d;\mathbb{R}^d)}$ be the associated supremum norm. Also we define $\mathcal{P}\left(\mathbb{T}^d\right)$ as the collection of all probability measure on $\mathbb{T}^d$.

Approach

Consider the Markov diffusion process $X_t^{x,\epsilon,\delta} \in \mathbb{R}^d$ governed by the operator :

$$\mathbf{L}_{\epsilon,\delta} = \frac{}{2}\sum_{i,j=1}^{d}\left(\quad {}^{*}\right)_{ij}\;\frac{x}{}\;\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} B^{\epsilon,\delta}\;\frac{x}{}\;\frac{\partial}{\partial x_i}$$

The trajectories of this process can be constructed with the help of the SDE:

$$\begin{cases} dX_t^{x,\epsilon,\delta} = \sqrt{\quad}\;\;\dfrac{X_t^{x,\epsilon,\delta}}{\delta}\;\; dW_t + B^{\epsilon,\delta}\;\;\dfrac{X_t^{x,\epsilon,\delta}}{\delta}\;\; dt \\ X_0^{x,\epsilon,\delta} = x \end{cases}$$

assumption

$: \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ and $B^{\epsilon,\delta} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ are regular applications. The vector-valued function $B^{\epsilon,\delta}$ is given by :

$$B^{\epsilon,\delta} = -B_0 + B_1 + B_2^{\epsilon,\delta}$$

$B_0; B_1$ and $B_2^{\epsilon,\delta}$ are $C^\infty \ \ \mathbb{R}^d; \mathbb{R}^d$ for all $\ ; \ \ > 0$ and

$$\lim_{\epsilon,\delta\to 0} \| B_2^{\epsilon,\delta} \|_{C_p(\mathbb{R}^d,\mathbb{R}^d)} = 0$$

hypothesis

Since the two parameters (homogenization) and (large deviation) tend to zero, we consider a new defined parameter

$$_\epsilon =$$

Suppose that $\lim_{\epsilon \downarrow 0} \frac{\delta_\epsilon}{\epsilon} = \gamma$, where > 0 a constant . There results that the homogenization parameter and the large deviation parameter go at the same rate

some rescalings

covering map

Define $\tilde{X}_t^{x,\epsilon,\delta_\epsilon}$ by : $\boldsymbol{\tilde{X}_t^{x,\epsilon,\delta_\epsilon}} = \frac{1}{\delta_\epsilon} X^{x,\epsilon,\delta_\epsilon}_{\left(\frac{\delta_\epsilon}{\sqrt{\epsilon}}\right)^2 t}$

Then

$$\begin{cases} d\tilde{X}_t^{x,\epsilon,\delta} = \quad \tilde{X}_t^{x,\epsilon,\delta} \quad d\tilde{W}_t + \frac{\delta_\epsilon}{\epsilon} B^{\epsilon,\delta} \quad \tilde{X}_t^{x,\epsilon,\delta} \quad dt \\ \tilde{X}_0^{x,\epsilon,\delta} = \frac{x}{\delta_\epsilon} \end{cases}$$

where $\tilde{W}_t^{\epsilon,\delta_\epsilon} = \frac{\sqrt{\epsilon}}{\delta_\epsilon} W^{\epsilon,\delta_\epsilon}_{\left(\frac{\delta_\epsilon}{\sqrt{\epsilon}}\right)^2 t}$ is a Brownian motion.

new consideration

the torus $\mathbb{T}^d$

Thereafter, consider the process $\left\{ \tilde{X}_t^{\epsilon,\delta_\epsilon} : t \geq 0 \right\}$ $\mathbb{T}^d$-valued which generator is defined by :

$$\tilde{L}_{\epsilon,\delta_\epsilon} = \frac{1}{2} \sum_{i,j=1}^{d} (\quad {}^{*})_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\epsilon}{\quad} \sum_{i=1}^{d} B^{\epsilon,\delta_\epsilon}(x) \frac{\partial}{\partial x_i}$$

Consider $a = \quad {}^{*}$, the above generator converges to the operator

$$L_\gamma = \frac{1}{2} \sum_{i,j=1}^{d} (a)_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} B_i^0(x) \frac{\partial}{\partial x_i} + \quad \sum_{i=1}^{d} B_i^1(x) \frac{\partial}{\partial x_i}$$

reference

tools

The basic set of calculations of this subject involves deriving the Varadhan formula and in identifying the constant $\overline{C}_2$. The main technique for showing that is the following result : (Baxendale and Stroock (1988), corollary 1.12, p[183] to p[185])

LDP

rate function

Let $\overline{\mathcal{J}}^2(\ )$ defined by Freidlin and Sowers (1999), the limit of

$$\log \mathbb{E}\left[\exp\left(\frac{1}{\ }\langle\ ; X^{x,\epsilon,\delta_\epsilon}\rangle\right)\right]$$

$$\overline{\mathcal{J}}^2(\theta) = \inf_{\{\phi \in C^\infty(\mathbb{T}^d)\}} \sup_{\{\mu \in \mathcal{P}(\mathbb{T}^d)\}} \int_{\mathbb{T}^d} \left(\frac{1}{2} \sum_{k=1}^{d} \langle (I - \nabla\phi)\,\sigma_k(z), \theta\rangle^2 + \langle (I - \nabla\phi)\, B_1(z), \theta\rangle \right.$$
$$\left. + \frac{1}{\gamma}\langle (B_0 - L_0\phi)(z), \theta\rangle \right) \mu(dz)$$

Define the Legendre-fenchel of $\overline{\mathcal{J}}^2(\ )$

$$\mathcal{J}^2(\theta) = \sup_{\{\theta' \in \mathbb{R}^d\}} \Big\langle\ ;\ {}'\Big\rangle - \overline{\mathcal{J}}^2(\ ')$$

large deviation principle

LDP

Define in addition :

$$\mathbf{S}^2_{0,T}(\phi) = \begin{cases} \int_0^T \mathcal{J}^2\left(\dot{} (s)\right) ds & \text{if absolutely continuous and } (0) = x \\ \infty & \text{if not} \end{cases}$$

Theorem (Freidlin and Sowers 1999)

Fix $T > 0$ and $x \in \mathbb{R}^d$. The family $\left\{ X_t^{x,\epsilon,\delta_\epsilon} : 0 \leq t \leq T \right\}_{\epsilon>0}$ of $C\left([0;T];\mathbb{R}^d\right)$-valued random variables has a large deviation principle with rate function $S^2_{0,T}(\ )$ for all $\in C\left([0;T];\mathbb{R}^d\right)$.

barrier

constant

Now define

$$\overline{C}_2 = \sup_{\mu\in\mathcal{P}(\mathbb{T}^d)} \int_{\mathbb{T}^d} c(z)\ (dz)$$

fundemental theorem

Varadhan formula

Theorem

Let c be an element of $C^\infty\left(\mathbb{T}^d\right)$ and D a Borel subset of $C\left([0;t];\mathbb{R}^d\right)$. Then

$$\liminf_{\epsilon\downarrow 0} \log \mathbb{E}\left[\mathbf{1}_D\left(X_t^{x,\epsilon,\delta_\epsilon}\right) e^{\left\{\frac{1}{\epsilon}\int_0^t c\left(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon}\right) ds\right\}}\right] \geq t\overline{C}_2 - \inf_{\left\{\phi\in\overset{\circ}{D}\right\}} S_{0,t}^2(\ )$$

$$\limsup_{\epsilon\downarrow 0} \log \mathbb{E}\left[\mathbf{1}_D\left(X_t^{x,\epsilon,\delta_\epsilon}\right) e^{\left\{\frac{1}{\epsilon}\int_0^t c\left(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon}\right) ds\right\}}\right] \leq t\overline{C}_2 - \inf_{\left\{\phi\in\overline{D}\right\}} S_{0,t}^2(\ )$$

behavior of $u^{;\epsilon}$

Feynman Kac

$$\boldsymbol{u}^{\epsilon,\delta_\epsilon}(\boldsymbol{t},\boldsymbol{x}) = \mathbb{E}\left[g\left(X_t^{x,\epsilon,\delta_\epsilon}\right) e^{\frac{1}{\epsilon}\int_0^t c\left(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon}, Y_s^{x,\epsilon,\delta_\epsilon}\right) ds} \right]$$

BSDE

Pardoux Peng 1992

$$\left\{\begin{array}{l} Y_s^{x,\epsilon,\delta_\epsilon} = g\left(X_t^{x,\epsilon,\delta_\epsilon}\right) + \frac{1}{\epsilon}\int_s^t f\left(\frac{X_r^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon}, Y_r^{x,\epsilon,\delta_\epsilon}\right) dr - \frac{1}{\sqrt{\epsilon}}\int_s^t Z_r^{x,\epsilon,\delta_\epsilon} dW_r \\ \mathbb{E}\left\{\int_s^t \left|Z_r^{x,\epsilon,\delta_\epsilon}\right|^2 dr\right\} < \infty \end{array}\right. ; 0 \leq s \leq t$$

and

$$Y_0^{x,\epsilon,\delta_\epsilon} = u^{\epsilon,\delta_\epsilon}(t;x)$$

viscosity solution

Pradeilles 1998

Since $u^{\epsilon,\delta_\epsilon}(t;x) > 0$, denote $\boldsymbol{v^{\epsilon,\delta_\epsilon}(t,x)} = \log u^{\epsilon,\delta_\epsilon}(t;x)$ and observe, that $v^{\epsilon,\delta_\epsilon}(t;x)$ is a viscosity solution of :

$$\begin{cases} \dfrac{\partial v^{\epsilon,\delta_\epsilon}}{\partial t}(t,x) &= L_{\epsilon,\delta_\epsilon} v^{\epsilon,\delta_\epsilon}(t,x) + \dfrac{1}{2}\left\|\nabla v^{\epsilon,\delta_\epsilon}(t,x)\sigma(x)\right\|^2 + c\left(\dfrac{x}{\delta_\epsilon}, u^{\epsilon,\delta_\epsilon}(t,x)\right) \\ v^{\epsilon,\delta_\epsilon}(0,x) &= \epsilon \log\left(g(x)\right), \qquad x \in G_0 \\ \lim\limits_{t\to 0} v^{\epsilon,\delta_\epsilon}(t,x) &= -\infty, \qquad x \in \mathbb{R}^d \backslash G_0 \end{cases}$$

some notation

definition

Let us define a distance in $\mathbb{R}^+ \times \mathbb{R}^d$, by for $(t;x);(s;y) \in \mathbb{R}^+ \times \mathbb{R}^d$:

$$\boldsymbol{d}\left\{(\boldsymbol{t},\boldsymbol{x}),(\boldsymbol{s},\boldsymbol{y})\right\} = \max\left\{|t-s|;\|x-y\|\right\}$$

Let us now introduce some notation :

$$\boldsymbol{A} = \inf_{\left\{\phi\in C^\infty\left(\mathbb{T}^d\right)\right\}} \sup_{\mu\in\mathcal{P}\left(\mathbb{T}^d\right)} \int_{\mathbb{T}^d} (I-\nabla\quad)\,a\,(I-\nabla\quad)(z)\ \ (dz)$$

$$\boldsymbol{B} = \inf_{\left\{\phi\in C^\infty\left(\mathbb{T}^d\right)\right\}} \sup_{\left\{\mu\in\mathcal{P}\left(\mathbb{T}^d\right)\right\}} \int_{\mathbb{T}^d} \left[(I-\nabla\quad)\,B_1 + \frac{1}{\gamma}\left(B_0 - L_0\quad\right)\right](z)\ \ (dz)$$

$$\rho^2\left(\boldsymbol{t},\boldsymbol{x},\boldsymbol{G_0}\right) = \inf_{\left\{\phi\in C\left([0,t],\mathbb{R}^d\right);\phi_0=x;\phi_t\in G_0\right\}} S_2^{0,\tau}(\quad)$$

asymptotic behavior

Define

$$\overline{\boldsymbol{v}}(\boldsymbol{t},\boldsymbol{x}) = \limsup_{\eta\to 0} \quad v^{\epsilon,\delta_\epsilon}(s;y) : \ (s;y) \in \mathcal{B}\{(t;x); \quad \}$$

$$\underline{\boldsymbol{v}}(\boldsymbol{t},\boldsymbol{x}) = \liminf_{\eta\to 0} \quad v^{\epsilon,\delta_\epsilon}(s;y) : \ (s;y) \in \mathcal{B}\{(t;x); \quad \}$$

Theorem

$\underline{v}$ and $\overline{v}$ are sub and super viscosity solutions of :

$$\begin{cases} \max_w \left(\frac{\partial w}{\partial t}(t,x) - \frac{1}{2}\langle A\nabla w(t,x), \nabla w(t,x)\rangle - \langle B, \nabla w(t,x)\rangle - \overline{C}_2 \right) = 0, \ x \in \mathbb{R}^d, t > 0 \\ w(0,x) \ = 0, \qquad x \in G_0 \\ \lim_{t\to 0} w(t,x) \ = \ -\infty, \qquad x \in \mathbb{R}^d \setminus G_0 \end{cases}$$

asymptotic behavior

definition

Let $\mathcal{O}$ is open subset in $\mathbb{R}^+ \times \mathbb{R}^d$, define the function on $[0\,;\infty[\ \times C\ \ [0\,;\infty[\ \times \mathbb{R}^d$

$$\tau = \ (t;\ \ ) = \inf\{s : (t - s;\ \ (s)) \in \mathcal{O}\}$$

Take $\boldsymbol{\Theta}_t$ the set of Markov functions .
Use the process

$$\boldsymbol{V(t,x)} = \inf_{\tau \in \Theta_t} \left(\overline{C}_2 \ - \inf_{\{\phi \in C([0,t],\mathbb{R}^d);\phi_0 = x;\phi_t \in G_0\}} S_2^{0,\tau}(\ ) \right)$$

asymptotic behavior

using the fact (Pradeilles 1998):

$$-\ {}^{2}(t; x; G_0) \leq \underline{v}(t; x) \leq \overline{v}(t; x) \leq \min\ \ Kt -\ {}^{2}(t; x; G_0)\, ; 0$$

Theorem

For $(t; x) \in \mathbb{R}_+^* \times \mathbb{R}^d$,

$$\lim_{\epsilon \downarrow 0}\ \log u^{\epsilon, \delta_\epsilon}(t; x) = V(t; x)$$

asymptotic behavior

corollary

Let $\mathcal{M}$ and $\mathcal{E}$ be a partition of $\mathbb{R}^+ \times \mathbb{R}^d$, such that:

$$\mathcal{M} = \left\{ (t;x) \in \mathbb{R}^+ \times \mathbb{R}^d \ : \ V(t;x) = 0 \right\}$$
$$\mathcal{E} = \left\{ (t;x) \in \mathbb{R}^+ \times \mathbb{R}^d \ : \ V(t;x) < 0 \right\}$$

Theorem

We have

$$\lim_{\epsilon \downarrow 0} u^{\epsilon,\delta_\epsilon(t,x)} = \begin{cases} 0 \ \textit{uniformly from any compact} \ \mathcal{K} \ \textit{of} \ \mathcal{E} \\ 1 \ \textit{uniformly from any compact} \ \mathcal{K}' \ \textit{of} \ \mathring{\mathcal{M}} \end{cases}$$