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Chapter 1

Generalities on stochastic processes

1.1 Probability spaces an random variables

A probability space is a triplet $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is a set, $\mathcal{F}$ is a $\sigma-$algebra on Ω, and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ that is a positive measure with mass one. This mathematical object is a model for a random experiment whose outcome cannot be exactly told in advance. The set Ω stands for the collection of all possible outcomes of the experiment. The $\sigma-$algebra $\mathcal{F}$ is the collection of all events that may occur during the realization of the experiment. An event F is said to occur if the outcome of the experiment happens to belong to F. The number $\mathbb{P}(F)$ is called the probability of the event F and it is the quantification of the chance of occurrence of the event F.

Let $(E, \mathcal{E})$ be a measurable space. A random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $(E, \mathcal{E})$ is a mapping $X : \Omega \longrightarrow E$ which is measurable relative to $\mathcal{F}$ and $\mathcal{E}$, that is $X^{-1}(A) = \{\omega \in \Omega : X(\omega) \in A\}$ is an event for all $A \in \mathcal{E}$.

Let X be a $E-$valued random variable. The image of the measure $\mathbb{P}$ by X is called the probability law or the distribution of X and is denoted by $\mathbb{P}_X$. Thus,

$$\mathbb{P}_X(A) = \mathbb{P}(X^{-1}(A)) = \mathbb{P}(X \in A), \text{ for all } A \in \mathcal{E}.$$

The random variables $X_1, \ldots, X_n$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values respectively in $(E_1, \mathcal{E}_1), \ldots, (E_n, \mathcal{E}_n)$ are said to be independent if the probability law of the $n-$uplet $(X_1, \dots, X_n)$ is the product measure $\otimes_i^n \mathbb{P}_{Xi}$ that is for all $A_i \in \mathcal{E}_i$, $i = 1, \dots, n$

$$\mathbb{P}(X_1 \in A_1, \dots, X_n \in A_n) = \prod_{i=1}^{n} \mathbb{P}(X_i \in A_i).$$

Thus, the random variables $X_1, \dots, X_n$ are independent means that the $\sigma-$algebras $\sigma(X_i)$ generated respectively by X_i are independent that is for any event $A_i \in \sigma(X_i)$, $i = 1, \dots, n$ the events $A_1, \dots, A_n$ are independent.

1.2 Stochastic processes

Let $\mathbb{T}$ be a set, for example $\mathbb{T} = \mathbb{N}, \mathbb{R}, \mathbb{R}_+, \mathbb{R}^d$. A collection $(X_t)_{t \in \mathbb{T}}$ of random variables taking values in $(E, \mathcal{E})$ is called a stochastic process with state space $(E, \mathcal{E})$ and parameter set $\mathbb{T}$.

Let $E^{\mathbb{T}}$ be the set of all maps from $\mathbb{T}$ to E. $E^{\mathbb{T}}$ is endowed with the product $\sigma-$algebra $\bigotimes_{t \in \mathbb{T}} \mathcal{E}$. The map

$$\begin{array}{rccc} X: & \Omega & \longrightarrow & E^{\mathbb{T}} \\ & \omega & \longmapsto & (X_t(\omega))_{t \in \mathbb{T}} \end{array} \tag{1.1}$$

is a random variable taking values in $(E^{\mathbb{T}}, \bigotimes_{t \in \mathbb{T}} \mathcal{E})$.

For any $\omega \in \Omega$, $X(\omega)$ is called a trajectory of the stochastic process $(X_t)_{t \in \mathbb{T}}$. The distribution of the map X is called the probability law of the stochastic process $(X_t)_{t \in \mathbb{T}}$ and is denoted by $\mathbb{P}_X$

Let

$$\mathcal{C} = \left\{ \prod_{t \in \mathbb{T}} A_t \text{ with } A_t \in \mathcal{E} \text{ and } \mathrm{Card}\{t \in \mathbb{T} : A_t \neq E\} < \infty \right\}$$

denote the set of finite dimensional cylinders of $E^{\mathbb{T}}$.

Since $\bigotimes_{t \in \mathbb{T}} \mathcal{E}$ is generated by $\mathcal{C}$ and $\mathcal{C}$ is stable by finite intersection, the monotone class theorem implies that the probability law of the stochastic process $(X_t)_{t \in \mathbb{T}}$ is determined by its values on $\mathcal{C}$. For any cylinder $C = \prod_{t \in \mathbb{T}} A_t$ there exists a finite number of indexes $t_1, \dots, t_n$ such that $A_{t_i} \neq E$, $i = 1, \dots, n$. Therefore

$$\mathbb{P}_X(C) = \mathbb{P}\left((X_{t_1}, \cdots, X_{t_n}) \in A_{t_1} \times \cdots \times A_{t_n}\right).$$

Thus, the probability law of the stochastic process $(X_t)_{t\in\mathbb{T}}$ is determined by its finite dimensional probability laws that is probability laws of random vectors $(X_{t_1},\cdots,X_{t_n})$, $t_1,\ldots,t_n\in\mathbb{T}$.

A stochastic process may also be regarded as a map denoted again by X which is defined on $\Omega\times\mathbb{T}$ and takes values in E. More precisely

$$\begin{array}{rccc} X: & \Omega\times\mathbb{T} & \longrightarrow & E \\ & (\omega,t) & \longmapsto & X_t(\omega). \end{array} \tag{1.2}$$

Let $(X_t)_{t\in\mathbb{T}}$ and $(Y_t)_{t\in\mathbb{T}}$ be two $E-$valued stochastic processes defined on $(\Omega,\mathcal{F},\mathbb{P})$. Then, one is said to be a modification of the other if, for any $t\in\mathbb{T}$

$$\mathbb{P}(X_t=Y_t)=1.$$

They are said to be indistinguishable if

$$\mathbb{P}(X_t=Y_t \text{ for all } t\in\mathbb{T})=1.$$

Remark 1 *If two stochastic processes are indistinguishable then there are modifications of each other. In general, the inverse implication is not true but if $\mathbb{T}$ is countable then it holds.*

From now on, $\mathbb{T}\subset\mathbb{R}$. If the map X in (1.2) is measurable relative to $\mathcal{F}\otimes\mathcal{B}(\mathbb{T})$ and $\mathcal{E}$, where $\mathcal{B}(\mathbb{T})$ stands for the Borel $\sigma-$algebra on $\mathbb{T}$, we say that the stochastic process $(X_t)_{t\in\mathbb{T}}$ is measurable.

Let $(X_t)_{t\in\mathbb{T}}$ be a $\mathbb{R}^d-$valued stochastic process. If for any $\omega\in\Omega$ the trajectory

$$\begin{array}{rccc} X(\omega): & \mathbb{T} & \longrightarrow & \mathbb{R}^d \\ & t & \longmapsto & X_t(\omega) \end{array}$$

is continuous, we say that $(X_t)_{t\in\mathbb{T}}$ is a continuous stochastic process.

Remark 2 *If two stochastic processes are continuous and modifications of each other then they are indistinguishable.*

Let $\mathcal{C}=\mathcal{C}(\mathbb{T},\mathbb{R}^d)$ stands for the set of $\mathbb{R}^d-$valued continuous functions defined on $\mathbb{T}$. So, the trajectories of a $\mathbb{R}^d-$valued continuous stochastic process are in $\mathcal{C}$. If $\mathcal{C}$ is endowed with the topology of uniform convergence on compact subsets of $\mathbb{T}$ then the map in (1.1) is measurable relative to $\mathcal{F}$ and $\mathcal{B}(\mathcal{C})$ where $\mathcal{B}(\mathcal{C})$ is the Borel $\sigma-$algebra of $\mathcal{C}$ and the probability law of the stochastic process $(X_t)_{t\in\mathbb{T}}$ is a probability measure on $(\mathcal{C},\mathcal{B}(\mathcal{C}))$.

1.3 Gaussian spaces and Gaussian processes

Definition 3 *A Gaussian space is a closed linear subset of* $L^2(\Omega, \mathcal{F}, \mathbb{P})$ *formed by centered Gaussian random vectors.*

For example if $X = (X_1, \cdots X_n)$ is a $\mathbb{R}^d-$valued centered Gaussian vector then the linear subset generated by $\{X_1, \cdots, X_n\}$ is a Gaussian space.

Definition 4 *A real-valued stochastic process* $(X_t)_{t\in\mathbb{T}}$ *is a Gaussian process if its finite dimensional probability laws are Gaussian, that is, for any* $t_1, \ldots, t_n \in \mathbb{T}$, $(X_{t_1}, \cdots, X_{t_n})$ *is a Gaussian random vector.*

If $(X_t)_{t\in\mathbb{T}}$ is a Gaussian process then the linear set generated by the coordinates of $(X_t)_{t\in\mathbb{T}}$,

$$Vec(X) = \left\{ \sum_{i=1}^{n} u_i X_{t_i};\ n \geq 1,\ u_i \in \mathbb{R},\ t_i \in \mathbb{T},\ i \leq n \right\}$$

is formed by Gaussian random variables.

Proposition 5 *Let* $(X_n)_{n\geq 0}$ *be a sequence of real-valued Gaussian random variables with* $\mathbb{E}(X_n) = m_n$ *and* $var(X_n) = \sigma_n^2$. *Assume that* $(X_n)_{n\geq 0}$ *converges in law to a random variable* X. *Then*

i) X *is a real-valued Gaussian random variable with* $\mathbb{E}(X) = m = \lim_{n\to\infty} m_n$ *and* $var(X) = \sigma^2 = \lim_{n\to\infty} \sigma_n^2$.

ii) *If* $(X_n)_{n\geq 0}$ *converges in probability to* X, *then the convergence holds in* L^p, $1 \leq p < \infty$.

Proof. Exercise. ■

In view of Proposition 5, $\overline{Vec(X)}^{L^2}$ the closure in L^2 of $Vec(X)$ is a closed linear subset of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ whose elements are also Gaussian random variables.

Definition 6 *Let* $X = (X_t)_{t\in\mathbb{T}}$ *be a Gaussian process. The Gaussian space*

$$H^X = \overline{Vec(X_t - \mathbb{E}(X_t);\ t \in \mathbb{T})}^{L^2}$$

is said to be associated to the stochastic process X.

Remark 7 *If the stochastic process X is centered then H^X is the Gaussian space generated by the coordinates of X, that is, the smallest Gaussian space enclosing all coordinates of X.*

Exercise 8 *Let H be a Gaussian space and let $(H_i)_{i\in I}$ be a family of linear subset of H. Prove that the subsets $H_i,\ i \in I$ are orthogonal if and only if the $\sigma-$algebras $\sigma(H_i)$, $i \in I$ are independent.*

1.4 Gaussian measures

Definition 9 *Let μ be a $\sigma-$finite measure on a measurable space $(E, \mathcal{E})$. A Gaussian measure on $(E, \mathcal{E})$ of intensity μ is an isometry G from $L^2(E, \mathcal{E}, \mu)$ onto a Gaussian space.*

For any $f \in L^2(E, \mathcal{E}, \mu)$, $G(f)$ is a centered Gaussian random variable with variance

$$\mathbb{E}\left[(G(f))^2\right] = \|G(f)\|_{L^2(\Omega,\mathcal{F},\mathbb{P})} = \|f\|_{L^2(E,\mathcal{E},\mu)}.$$

In particular if $A \in \mathcal{E}$ with $\mu(A) < \infty$ then the law of $G(\mathbf{1}_A)$ is $\mathcal{N}(0, \mu(A))$.

From now on we put $G(A) = G(\mathbf{1}_A)$.

Exercise 10 *Let μ be a $\sigma-$finite measure on a measurable space $(E, \mathcal{E})$.*

1) *Prove that there exists a Gaussian measure on $(E, \mathcal{E})$ of intensity μ.*

2) *Let G be a Gaussian measure of intensity μ.*

a) *Prove that if $A_1, \dots, A_n \in \mathcal{E}$ are disjointed with $\mu(A_i) < \infty$, $\ i = 1, \dots, n$ then $G(A_1), \dots, G(A_n)$ are independent Gaussian random variables.*

b) *Let $A \in \mathcal{E}$ with $\mu(A) < \infty$ such that there exists a countable partition $A_1, A_2, \dots$ of A in elements of $\mathcal{E}$. Prove that the serie $\sum_{i=1}^{\infty} G(A_i)$ is convergent in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and*

$$G(A) = \sum_{i=1}^{\infty} G(A_i), \quad a.s.$$

Chapter 2

Brownian Motion

2.1 Brief historical

The Brownian motion was discovered in 1822 by the British botanist Robert Brown. Indeed, observing the displacement of grains of pollen suspended in a liquid, he noticed that the movement of the grains were very irregular . This irregularity is due to collisions of the grains with the particles of the liquid leading to a dispersion (diffusion) of the grains.

In 1900, Bachelier interested in the fluctuations of stock price published the first quantitative work on Brownian motion. Einstein(1905) studying molecular dynamics hightlighted the Brownian motion. He derived the transition density. A rigorous mathematical study of Brownian motion began with the works of Wiener (1923, 1924) who has given the first existence proof. One of the deep study of Brownian motion is due to P. Lévy (1939, 1948) who provided a construction by interpolation and derived some fine properties of first passage times and sample paths of Brownian motion.

Nowadays numerous works can be found in literature on Brownian motion for example the stochastic calculus of K. Itô (1958) which is a very useful tools in various fields such as finance, economic, biology, partial differential equations, etc.

2.2 Definition and some properties

Definition 11 *A stochastic process* $B = (B_t)_{t\geq 0}$ *with state space* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *is called a Brownian motion if it satisfies:*

(I) *B is a continuous process*

ii) $B_0 = 0,$ *a.s.*

iii) *B has independent increments that is for any* $0 \le t_0 < \cdots < t_n$, *the increments* $B_{t_0},\ B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}}$ *are independent random variables*

iv) *For any* $s < t$, $B_t - B_s$ *has the Gaussian distribution* $\mathcal{N}(0, t-s)$.

Remark 12 *a) Condition (iv) implies that Brownian motion has stationary increments which means that the law of* $B_t - B_s$ *depends on* t *and* s *only through the difference* $t - s$.
b) Condition (iii) is equivalent to the following one: for any $s < t$, $B_t - B_s$ *is independent of the* $\sigma-$*algebra generated by* $\{B_u : u \le s\}$ *which is denoted by* $\mathcal{F}_s^B$.

Proposition 13 *A stochastic process B is a Brownian motion if and if B is a real-valued continuous centered Gaussian process with covariance function*

$$\mathbb{E}(B_t B_s) = \min(s,t) \tag{2.1}$$

.

Proof. Assume that B is a Brownian motion. So, B is a continuous process. Since the increments of B are independent and have Gaussian distributions, by using a transformation of random variables argument one can prove that for any $0 = t_0 < \cdots < t_n$, $(B_{t_1}, \ldots, B_{t_n})$ has a Gaussian distribution with density function

$$f(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{t_1(t_2 - t_1)\cdots(t_n - t_{n-1})}} \times \exp\left(-\frac{1}{2}\sum_{i=1}^{n} \frac{(x_i - x_{i-1})^2}{t_i - t_{i-1}}\right)$$

where $x_0 = 0$.

Since if $(X_1, \ldots, X_n)$ is a Gaussian Random variable then for any permutation σ of $\{1, \ldots, n\}$ the random variable $(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$ is again a Gaussian random variable, it follows that for any $t_1, \ldots, t_n \in \mathbb{R}_+$, $(B_{t_1}, \ldots, B_{t_n})$ is a Gaussian random variable.

In view of (iv) and (ii), $\mathbb{E}(B_t) = 0$. Thus, B is a continuous centered Gaussian process.

For any $s < t$,

$$\begin{aligned} \mathbb{E}(B_s B_t) &= \mathbb{E}(B_s(B_t - B_s) + B_s^2) \\ &= \mathbb{E}(B_s(B_t - B_s)) + \mathbb{E}(B_s^2) \\ &= \mathbb{E}(B_s)\mathbb{E}(B_t - B_s) + s \\ &= 0 + s \\ &= \min(s,t) \end{aligned}$$

The two last lines follow from property (iii) and (iv).

Reciprocally, Assume that B is a continuous Gaussian process with the covariance function given by (2.1) . Then $\mathbb{E}(B_0^2) = 0$ and for any $s < t < u$

$$\begin{aligned} \mathbb{E}((B_t - B_s)^2) &= \mathbb{E}(B_t^2) + \mathbb{E}(B_t^2) - 2\mathbb{E}(B_s B_t) \\ &= t\ + s - 2\min(s,t) \\ &= t - s \end{aligned}$$

and

$$\mathbb{E}((B_t - B_s)(B_u - B_t)) = 0.$$

Therefore B satisfies conditions (i)-(iv) of Definition 11 ■

Proposition 14 *Assume that B is a Brownian motion. Then the following hold:*

a) *(Symmetry) The stochastic process $(-B_t)_{t\geq 0}$ is a Brownian motion.*

b) *(Scaling) $X = (a^{-1/2}B_{at})_{t\geq 0}$ is a Brownian motion for any fixed real number $a > 0$.*

c) *(Time inversion) Let us put $Y_0 = 0$ and $Y_t = tB_{1/t}$ for $t > 0$. $Y = (Y_t)_{t\geq 0}$ is a Brownian motion.*

d) *For any $t_0 \geq 0$, $B^{(t_0)} = (B_{t+t_0} - B_{t_0})_{t\geq 0}$ is a Brownian motion independent of the $\sigma-$algebra $\mathcal{F}_{t_0}^B$.*

e) *(Reversibility) For any fixed $T > 0$, $Z = (B_{T-t} - B_T)_{t\in[0,T]}$*

Proof. It is not difficult to prove that $-B$, X, $(Y_t)_{t>0}$, $B^{(t_0)}$ and Z are centered continuous Gaussian processes with covariance functions given by (2.1). It remains to prove the continuity of Y at 0 which is equivalent to

$$\lim_{t\to\infty} \frac{B_t}{t} = 0, \ a.s.$$

Let $n \geq 0$ be such that $n < t \leq n+1$, we have

$$\begin{aligned} \left|\frac{B_t}{t}\right| &\leq \left|\frac{B_n}{n}\right| + \frac{1}{n}\left|B_t - B_n\right| \\ &\leq \left|\frac{B_n}{n}\right| + \frac{1}{n}\sup_{0\leq s\leq 1}\left|B_{n+s} - B_n\right|. \end{aligned}$$

Since

$$B_n = \sum_{i=1}^{n}(B_i - B_{i-1}),$$

and the random variables $B_i - B_{i-1}$, $i \geq 1$ are independent and identically distributed, the strong law of large numbers implies that $\dfrac{B_n}{n}$ goes a.s. to 0 as $n \to \infty$.Now, by virtue of Kolmogorov's inequality (see [?]), for any $\varepsilon > 0$

$$\mathbb{P}\left(\frac{1}{n}\sup_{0\leq s\leq 1}\left|B_{n+s} - B_n\right| \geq \varepsilon\right) \leq \frac{1}{n^2\varepsilon^2}\mathbb{E}\left[(B_{n+1} - B_n)^2\right] = \frac{1}{n^2\varepsilon^2}.$$

Since $\sum 1/n^2$ is finite, Borel Cantelli lemma leads to $\frac{1}{n}\sup_{0\leq s\leq 1}|B_{n+s} - B_n$ goes a.s. to 0 as $n \to \infty$. Hence, $B_t/t \to 0$, a.s. as $t \to \infty$ which ends the proof. ■

2.3 Construction of Brownian motion

We begin by a theorem due to Kolmogorov and Čentsov (1956) which permits to prove the existence of a continuous version of a stochastic process.

Theorem 15 *Suppose that a stochastic process $X = (X_t)_{t\in[0,T]}$ satisfies the condition*

$$\mathbb{E}(|X_t - X_s|^{\alpha}) \leq C|t-s|^{1+\beta}, \quad 0 \leq s, t \leq T$$

for some positive constants α, β, and C. Then there exists a continuous modification $\widetilde{X} = (\widetilde{X}_t)_{t\in[0,T]}$ of X, which is locally Hölder-continuous with exponent γ for every $\gamma \in (0, \beta/\alpha)$, i.e.

$$\mathbb{P}\left(\omega : \sup_{\substack{0<t-s<h(\omega) \\ s,t\in[0,T]}} \frac{|\widetilde{X}_t(\omega) - \widetilde{X}_s(\omega)|}{|t-s|^\gamma} \leq \delta \right) = 1,$$

where $h(\omega)$ is an a.s. positive random variable and $\delta > 0$ is an appropriate constant.

Proof. See [**?**] ■

2.3.1 Canonical Brownian motion

The law of the Brownian motion is called Wiener measure which will be denoted μ_W. Let $\mathcal{C}$ stands for the space of continuous real-valued functions defined on $\mathbb{R}_+$. The Wiener measure is the probability measure on $(\mathcal{C}, \mathcal{B}(\mathcal{C}))$ caracterised by: for any $0 = t_0 < t_1 < \cdots < t_n$ and $A_0, A_1, \ldots, A_n$ in $\mathcal{B}(\mathbb{R})$

$$\begin{aligned} \mu_W(w \;\; \in \;\; & \mathcal{C} : (w(t_0), w(t_1), \ldots, w(t_n) \in A_0 \times A_1 \times \ldots \times A_n) = \\ & 1_{A_0}(0) \int_{A_1\times\ldots\times A_n} \frac{1}{(2\pi)^{n/2}\sqrt{t_1(t_2-t_1)\cdots(t_n-t_{n-1})}} \exp\left(-\frac{1}{2}\sum_{i=1}^n \frac{(x_i - x_{i-1})^2}{t_i - t_{i-1}}\right) dx_1 \cdots d \end{aligned}$$

Let us consider the canonical probability space $(\mathcal{C}, \mathcal{B}(\mathcal{C}), \mu_W)$ and define a stochastic process W by

$$W_t(w) = w(t), \quad t \geq 0.$$

This process is called the coordinate mapping process. It is not difficult to prove that W is a Brownian motion.

2.3.2 Construction via Gaussian measure

Let G be a Gaussian measure with intensity the Lebesgue measure on $\mathbb{R}_+$. let us put

$$B_t = G(([0,t]), \; t \geq 0.$$

It is not difficult to prove that B is a centered Gaussian process with covariance function given by (2.1). Now, we have

$$\mathbb{E}(|B_t - B_s|^4 = 3|t-s|^2, \text{ for any } s, t \geq 0.$$

Therefore in view of Theorem 15, there exists a continuous version of B which is a Brownian motion.

2.4 Path properties

Theorem 16 *(Blumenthal's zero-one law) Let $\mathcal{F}_{0^+}^B = \bigcap_{s>0} \mathcal{F}_s^B$. Then, every event in $\mathcal{F}_{0^+}^B$ has probability 0 or 1.*

Proof. Let $g : \mathbb{R}^k \longrightarrow \mathbb{R}$ be a bounded continuous function. For any $A \in \mathcal{F}_{0^+}^B$ and $0 < t_1 < \cdots < t_n$, in view of continuity argument and Lebesgue dominated convergence Theorem, we have

$$\mathbb{E}(1_A g(B_{t_1}, \ldots, B_{t_k})) = \lim_{\varepsilon \to 0} \mathbb{E}(1_A g(B_{t_1} - B_\varepsilon, \ldots, B_{t_k} - B_\varepsilon)).$$

If $\varepsilon < t_1$ then $B_{t_1} - B_\varepsilon, \ldots, B_{t_k} - B_\varepsilon$ are independent of the $\sigma-$algebra $\mathcal{F}_\varepsilon^B$. Since $\mathcal{F}_{0^+}^B \subset \mathcal{F}_\varepsilon^B$, these random variables are also independent of the $\sigma-$algebra $\mathcal{F}_{0^+}^B$. It follows that

$$\begin{aligned}
\mathbb{E}(1_A g(B_{t_1}, \ldots, B_{t_k})) &= \lim_{\varepsilon \to 0} \mathbb{E}(1_A g(B_{t_1} - B_\varepsilon, \ldots, B_{t_k} - B_\varepsilon)) \\
&= \lim_{\varepsilon \to 0} \mathbb{P}(A) \mathbb{E}(g(B_{t_1} - B_\varepsilon, \ldots, B_{t_k} - B_\varepsilon)) \\
&= \mathbb{P}(A) \mathbb{E}(g(B_{t_1}, \ldots, B_{t_k})).
\end{aligned}$$

Therefore, for every $t_1, \ldots, t_k > 0$, $\mathcal{F}_{0^+}^B$ is independent of $\sigma((B_{t_1}, \ldots, B_{t_k})$. Now, since $\sigma(B_t,\ t > 0)$ is generated by sets of the form $(B_{t_1} \in A_1, \ldots, B_{t_k} \in A_k)$, $t_i > 0$ and $A_i \in \mathcal{B}(\mathbb{R})$, $i = 1, \ldots, k$, we deduce that $\mathcal{F}_{0^+}^B$ is independent of $\sigma(B_t,\ t > 0)$. It is clear that $\sigma(B_t,\ t > 0) = \sigma(B_t,\ t \geq 0)$ because B_0 is the pointwise limit of B_t as $t \to 0$. Finally, since $\mathcal{F}_{0^+}^B \subset \sigma(B_t,\ t \geq 0)$ we derive that $\mathcal{F}_{0^+}^B$ is independent of himself that is every event in $\mathcal{F}_{0^+}^B$ is of probability 0 or 1. ■

Corollary 17 *We have a.s. , for any $\varepsilon > 0$,*

$$\sup_{0 \leq s \leq \varepsilon} B_s > 0, \qquad \inf_{0 \leq s \leq \varepsilon} B_s < 0.$$

Proof. Let (ε_p) be a sequence of positive real numbers decreasing to 0 and set

$$A_p = \left\{ \sup_{0 \le s \le \varepsilon_p} B_s > 0 \right\} \text{ and } A = \bigcap_p A_p.$$

Since (A_p) is a decreasing sequence we have $A \in \mathcal{F}^B_{0^+}$.

We have

$$\mathbb{P}(A_p) \ge \mathbb{P}(B_{\varepsilon_p} > 0) = \frac{1}{2}$$

$$\mathbb{P}(A) = \lim_{p \to \infty} \mathbb{P}(A_p).$$

Therefore, by virtue of Theorem 16, we conclude that $\mathbb{P}(A) = 1$.

It follows that for every $\varepsilon > 0$

$$\mathbb{P}\left(\sup_{0 \le s \le \varepsilon} B_s > 0 \right) = 1.$$

By using the symmetry of Brownian motion, we deduce that

$$\mathbb{P}\left(\inf_{0 \le s \le \varepsilon} B_s < 0 \right) = 1.$$

■

Exercise 18 *Prove the following assertions*

1) *For every* $\delta > 0$

$$\lim_{\delta \to 0} \mathbb{P}\left(\sup_{0 \le s \le 1} B_s > \delta \right) = 1$$

2) *For every* $\delta > 0$

$$\mathbb{P}\left(\sup_{0 \le s \le 1} B_s > \delta \right) = \mathbb{P}\left(\sup_{0 \le s \le 1/\delta^2} B_s > 1 \right)$$

3)

$$\mathbb{P}\left(\sup_{s \ge 0} B_s > 1 \right) = \mathbb{P}\left(\inf_{s \ge 0} B_s < -1 \right) = 1$$

4) *For every* $C > 0$

$$\mathbb{P}\left(\sup_{s\geq 0} B_s > C\right) = \mathbb{P}\left(\inf_{s\geq 0} B_s < -C\right) = 1$$

5) $\mathbb{P}(T_a < \infty) = 1$ *where for any* $a \in \mathbb{R}$, *we put* $T_a = \inf\{t \geq 0 : B_t = a\}$ *(with the convention inf* $\emptyset = +\infty$*).* T_a *is called the first passage time in* a.

6) *almost surely*

$$\limsup_{t\to\infty} B_t = +\infty, \quad \textit{and} \quad \liminf_{t\to\infty} B_t = -\infty.$$

Proposition 19 *(Quadratic variation of Brownian motion) Let* $0 = t_0^n < t_1^n < \cdots < t_{p_n}^n = t$ *a sequence of partition of of* $[0, t]$ *with mesh going to* 0 *that is* $\sup_{1\leq i\leq p_n}(t_i^n - t_{i-1}^n) \to 0$ *as* $n \to \infty$. *Then*

$$\lim_{n\to\infty} \sum_{i=1}^{p_n} (B_{t_i^n} - B_{t_{i-1}^n})^2 = t,$$

where the convergence holds in L^2.

Proof. Let $\Delta = \{t_0, t_1^n, \ldots, t_n^{p_n}\}$ be a partition of $[0, t]$. We put

$$V(\Delta) = \sum_{i=1}^{p_n} (B_{t_i^n} - B_{t_{i-1}^n})^2.$$

We have

$$\begin{aligned}
\mathbb{E}\left[(V(\Delta))^2\right] &= \sum_{i=1}^{p_n} \mathbb{E}\left[(B_{t_i^n} - B_{t_{i-1}^n})^4\right] + 2\sum_{1\leq i<j\leq p_n} \mathbb{E}\left[(B_{t_i^n} - B_{t_{i-1}^n})^2 (B_{t_j^n} - B_{t_{j-1}^n})^2\right] \\
&= 3\sum_{i=1}^{p_n} (t_i^n - t_{i-1}^n)^2 + 2\sum_{1\leq i<j\leq p_n} \left(t_i^n - t_{i-1}^n\right)\left(t_j^n - t_{j-1}^n\right) \\
&= 2\sum_{i=1}^{p_n} (t_i^n - t_{i-1}^n)^2 + \left[\sum_{i=1}^{p_n} (t_i^n - t_{i-1}^n)\right]^2 \\
&= 2\sum_{i=1}^{p_n} (t_i^n - t_{i-1}^n)^2 + t^2.
\end{aligned}$$

Now,

$$\mathbb{E}(V(\Delta)) = \sum_{i=1}^{p_n} (t_i^n - t_{i-1}^n) = t.$$

Therefore

$$\mathbb{E}\left[(V(\Delta) - t)^2\right] = 2\sum_{i=1}^{p_n} (t_i^n - t_{i-1}^n)^2 \leq 2 \sup_{1\leq i\leq p_n} (t_i^n - t_{i-1}^n)t.$$

The right hand side converges to 0 as $|\Delta| = \sup_{1\leq i\leq p_n}(t_i^n - t_{i-1}^n)$ goes to 0. ∎

Corollary 20 *The Brownain has no finite variation on any interval.*

Proof. Exercise. ∎

2.5 Brownian Motion and Martingales

Martingales are a very useful theory in the study of stochastic process. Particularly it is a fundamental notion in Itô's stochastic calculus. In this course, we only give an introduction of Martingale theory. The reader may see [**?**] for details.

2.5.1 Filtrations

Definition 21 *A filtration $\mathcal{F} = (\mathcal{F}_t : t \geq 0)$ is an increasing family of sub-$\sigma-$algebras of $\mathcal{A}$ that is for every $s \leq t$, $\mathcal{F}_s \subset \mathcal{F}_t$.*

Let $X = (X_t)_{t\geq 0}$ be a stochastic process. We put $\mathcal{F}_t^X = \sigma(X_s : s \leq t)$. Then $\mathcal{F}^X$ is a filtration called the natural filtration of X or the filtration generated by X.

Heuristically, we may regard a filtration $\mathcal{F}$ as a flow of information, with $\mathcal{F}_t$ representing all the information accumulated until time t. For example $\mathcal{F}_t^X$ has all the information regarding the past of X and the present X_t.

Let us set

$$\mathcal{F}_{t^+} = \bigcap_{s>t} \mathcal{F}_s.$$

It is clear that $(\mathcal{F}_{t^+} : t \geq 0)$ is a filtration which is finer that the filtration $(\mathcal{F}_t : t \geq 0)$ that is $\mathcal{F}_t \subset \mathcal{F}_{t^+}$ for every $t \geq 0$.

Heuristically, $\mathcal{F}_{t^+}$ has the same information as $\mathcal{F}_t$ plus the information just immediately after time t. For instance, let $X = (X_t)_{t\geq 0}$ a stochastic process representing the smooth random motion of a particle, that is $t \longrightarrow X_t(\omega)$ is a smooth function for every $\omega \in \Omega$. The $\sigma-$algebra $\mathcal{F}_{t^+}$ contained information by time t plus the velocity $V_t = \lim_{\varepsilon\to 0}(X_{t+\varepsilon} - X_t)/\varepsilon$, the acceleration and so on.

Definition 22 *A filtration $\mathcal{F} = (\mathcal{F}_t : t \geq 0)$ is said to be right-continuous if for every $t \geq 0$, $\mathcal{F}_t = \mathcal{F}_{t^+}$.*

It is not difficult to see that $(\mathcal{F}_{t^+} : t \geq 0)$ is a right-continuous filtration.

From now on $\mathcal{F} = (\mathcal{F}_t : t \geq 0)$ is a fixed filtration.

Definition 23 *A stochastic process $X = (X_t)_{t\geq 0}$ is $\mathcal{F}-$adapted or adapted relative to the filtration $\mathcal{F}$ if for every $t \geq 0$, the random variable X_t is $\mathcal{F}_t-$measurable.*

A stochastic process is adapted means that this process doesn't anticipate on the future. It is clear that any stochastic process is adapted relative to its natural filtration.

2.5.2 Stopping times

Definition 24 *A random time $\tau : \Omega \longrightarrow \overline{\mathbb{R}}_+$ is a $\mathcal{F}-$stopping time if for every $t \geq 0$ the event $\{\tau \leq t\}$ belongs to $\mathcal{F}_t$.*

Heuristically, a random time can be viewed as the time of occurrence of a particular event The information accumulated by time t is sufficient to decide if this event has occurred or not.

For instance let $X = (X_t)_{t\geq 0}$ be a real-valued, right-continuous and $\mathcal{F}-$adapted stochastic process. For every closed set F we put

$$\tau_F = \inf\{t \geq 0 : X_t \in F\}.$$

The random time τ_F called the hitting time of the set F is a $\mathcal{F}-$stopping time.

If we put $\sigma_F = \sup\{t \leq 1 : X_t \in F\}$ then σ_F is not a stopping time because for every $t \leq 1, \mathcal{F}_t$ is not sufficient to decide if $\sigma_F \leq t$ or not.

Every constant random time $\tau \equiv t_0$ is a stopping time and for every sequence $(\tau_n)_{n\geq 0}$ of stopping times, $\inf_n \tau_n$, $\sup_n \tau_n$ and $\lim_n \tau_n$ (if the limit exists) are stopping times.

For every $\mathcal{F}-$stopping time τ, we consider the set $\mathcal{F}_\tau = \{A \in \mathcal{A} : A \cap \{\tau \leq t\} \in \mathcal{F}_t\}$ called the past until τ. It is easy to prove that $\mathcal{F}_\tau$ is a $\sigma-$algebra and $\mathcal{F}_\tau \subset \mathcal{F}_\nu$ if ν is another $\mathcal{F}-$stopping time.

If $\tau \equiv t$ then $\mathcal{F}_\tau = \mathcal{F}_t$.

Exercise 25 *a) Let τ be a stopping time and V a random variable. Prove that V belongs to $\mathcal{F}_\tau$ if and only if $V\mathbf{1}_{\{\tau\leq t\}}$ is $\mathcal{F}_t-$measurable for every $t \in \overline{\mathbb{R}}_+$.*
b) Prove that if τ and ν are stopping times then $\mathcal{F}_\tau \cap \mathcal{F}_\nu = \mathcal{F}_{\tau\wedge\nu}$ and for any $\mathcal{F}_\tau-$measurable random variable V the following random variables are $\mathcal{F}_{\tau\wedge\nu}-$measurable:

$$V1_{\{\tau\leq\nu\}},\ V1_{\{\tau=\nu\}},\ V1_{\{\tau<\nu\}},$$

in particular the events $\{\tau \leq \nu\}$, $\{\tau = \nu\}$ and $\{\tau < \nu\}$ belong to $\mathcal{F}_{\tau\wedge\nu}$.

2.5.3 Martingales

Definition 26 *A stochastic process $M = (M_t)_{t\geq 0}$ is called an $\mathcal{F}-$submartingale ($\mathcal{F}-$supermartingale, $\mathcal{F}-$martingale) if*

i) *M is $\mathcal{F}-$adapted*

ii) *for every $t \geq 0$, M_t is integrable*

iii) *for every $s \leq t$,*

$$M_s \leq \mathbb{E}(M_t|\mathcal{F}_s) \quad (M_s \geq \mathbb{E}(M_t|\mathcal{F}_s), \quad M_s = \mathbb{E}(M_t|\mathcal{F}_s)).$$

Remark 27 *A stochastic process $M = (M_t)_{t\geq 0}$ is an $\mathcal{F}-$supermartingale if $-M$ is an $\mathcal{F}-$submartingale and an $\mathcal{F}-$martingale if it is both an $\mathcal{F}-$submartingale and an $\mathcal{F}-$supermartingale.*

Examples of martingales

1) Let $X = (X_t)_{t\geq 0}$ be an $\mathcal{F}-$adapted stochastic process with independent increments. If for every $t \geq 0$, X_t is integrable then $(X_t - \mathbb{E}(X_t))_{t\geq 0}$ is an $\mathcal{F}-$martingale. In particular a Brownian motion $B = (B_t)_{t\geq 0}$ is an $\mathcal{F}^B-$martingale.

2) Let $B = (B_t)_{t\geq 0}$ be a Brownian motion. $(B_t^2 - t)_{t\geq 0}$ is an $\mathcal{F}^B-$martingale. Indeed, for every $s \leq t$

$$\begin{aligned}
\mathbb{E}((B_t^2 - B_s^2)|\mathcal{F}_s) &= \mathbb{E}((B_t - B_s)^2 + 2B_tB_s - 2B_s^2|\mathcal{F}_s) \\
&= \mathbb{E}((B_t - B_s)^2|\mathcal{F}_s) + \mathbb{E}(2B_tB_s|\mathcal{F}_s) + \mathbb{E}(-2B_s^2|\mathcal{F}_s) \\
&= \mathbb{E}((B_t - B_s)^2) + 2B_s\mathbb{E}(B_t|\mathcal{F}_s) - 2B_s^2 \\
&= t - s + 2B_s^2 - 2B_s^2 \\
&= t - s.
\end{aligned}$$

3) Let $B = (B_t)_{t\geq 0}$ be a Brownian motion. For every $\theta \in \mathbb{R}$, put $B_t^\theta = \exp(\theta B_t - \frac{\theta^2}{2}t)$, $t \geq 0$. Then, B^θ is an $\mathcal{F}^B-$martingale and is called the exponential martingale. (Exercise)

It is easy to see from Jensen's inequality that if M is an $\mathcal{F}-$martingale then for every convex function $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\Phi(M_t)$ is integrable, $(\Phi(M_t))_{t\geq 0}$ is an $\mathcal{F}-$submartingale.

Theorem 28 *(Doob's stopping theorem). Let M be a right-continuous $\mathcal{F}-$martingale. For every bounded stopping times τ and ν such that $\tau \leq \nu$, M_τ and M_ν are integrable and*

$$M_\tau = \mathbb{E}(M_\nu|\mathcal{F}_\tau).$$

The equality being replaced by $\geq$ (resp. $\leq$) if M is an $\mathcal{F}-$supermartingale (resp. an $\mathcal{F}-$submartingale)

Proof. See ■

As a consequence of Doob's stopping Theorem, if M an $\mathcal{F}-$martingale then for every bounded stopping time τ,

$$\mathbb{E}(M_\tau) = \mathbb{E}(M_0). \qquad (2.2)$$

If τ is not bounded then (2.2) may be false. For instance, let us consider T_b the hitting time of a point b by a Brownian motion B that is $T_b = \inf\{t \geq 0 : B_t = b\}$. Since B has continuous trajectories, we have $B_{T_b} = b$. So $b = \mathbb{E}(B_{T_b}) \neq \mathbb{E}(B_0) = 0$. We will prove later that T_b is almost surely finite and not bounded.

There exist a general version of Doob's stopping theorem which doesn't need boundness of stopping times.

Theorem 29 *(Doob's stopping theorem for uniformly integrable martingale). Let M be a right-continuous $\mathcal{F}-$martingale and uniformly integrable. For every bounded stopping times τ and ν such that $\tau \leq \nu$, M_τ and M_ν are integrable and*

$$M_\tau = \mathbb{E}(M_\nu|\mathcal{F}_\tau).$$

Proof. see ■

Proposition 30 *Let $M = (M_t)_{t\geq 0}$ be an $\mathcal{F}-$martingale and τ a $\mathcal{F}-$stopping time. Then the stopped process $M^\tau = (M_{\tau\wedge t})_{t\geq 0}$ is an $\mathcal{F}-$martingale.*

Proof. First of all, let us remark that in view of Doob's stopping theorem, for every $s \leq t$, we have

$$\mathbb{E}(M_t^\tau|\mathcal{F}_{\tau\wedge s}) = M_s^\tau.$$

This equality proves that M^τ is a martingale relative to the filtration $\{\mathcal{F}_{\tau\wedge s} : s \geq 0\}$, but this is not what we want. Our aim is to prove that

$$\mathbb{E}(M_t^\tau|\mathcal{F}_s) = M_s^\tau. \tag{2.3}$$

For this fact, let us note that it is not difficult to prove that for any random variable Z,

$$1_{\{\tau\geq s\}}\mathbb{E}(Z|\mathcal{F}_{\tau\wedge s}) = 1_{\{\tau\geq s\}}\mathbb{E}(Z|\mathcal{F}_s).$$

Therefore, we have

$$1_{\{\tau\geq s\}}M_s^\tau = 1_{\{\tau\geq s\}}\mathbb{E}(M_t^\tau|\mathcal{F}_{\tau\wedge s}) = 1_{\{\tau\geq s\}}\mathbb{E}(M_t^\tau|\mathcal{F}_s). \tag{2.4}$$

Now,

$$\begin{aligned} 1_{\{\tau\leq s\}}\mathbb{E}(M_t^\tau|\mathcal{F}_s) &= \mathbb{E}(1_{\{\tau\leq s\}}M_t^\tau|\mathcal{F}_s) \\ &= \mathbb{E}(1_{\{\tau\leq s\}}M_s^\tau|\mathcal{F}_s) \\ &= 1_{\{\tau\leq s\}}M_s^\tau \end{aligned} \tag{2.5}$$

where we have used the equality $1_{\{\tau\leq s\}}M_t^\tau = 1_{\{\tau\leq s\}}M_s^\tau$ and the fact that $1_{\{\tau\leq s\}}M_s^\tau$ is $\mathcal{F}_s-$measurable.

Combining (2.4) and (2.5), we deduce (2.3). ■

Now, we deal with the strong Markov property of Brownian motion

Corollary 31 *For very $a, b > 0$, let us put*

$$\tau = \inf\{t \geq 0 : B_t \in]-a, b[\}.$$

We have

$$\mathbb{P}(T_b < T_{-a}) = \mathbb{P}(B_\tau = b) = \frac{a}{a+b}, \qquad \mathbb{P}(T_{-a} < T_b) = \mathbb{P}(B_\tau = -a) = \frac{b}{a+b}.$$

Proof. Let us put $M_t = B_t^2 - t$, and $\tau_n = \tau \wedge n$. Since τ_n is a bounded stopping time and M is a martingale, we have

$$\mathbb{E}(M_{\tau_n}) = \mathbb{E}(M_0) = 0.$$

It follows that

$$\mathbb{E}(B_{\tau_n}^2) = \mathbb{E}(\tau_n).$$

Now, since for every $n \geq 0, |\, B_{\tau_n}| \leq a \vee b$, we have

$$\mathbb{E}(\tau_n) \leq a^2 \vee b^2.$$

So by applying monotone convergence theorem, we derive that $\mathbb{E}(\tau) < \infty$ and therefore τ is a.s. finite.

By virtue of Doob's stopping Theorem, we have

$$\mathbb{E}(B_{\tau_n}) = \mathbb{E}(B_0) = 0.$$

By letting $n \to \infty$, Lebesgue dominated convergence theorem implies that

$$\mathbb{E}(B_\tau) = 0.$$

Therefore,

$$0 = \mathbb{E}(B_\tau) = b\mathbb{P}(B_\tau = b) - a\mathbb{P}(B_\tau = -a).$$

By combining $\mathbb{P}(B_\tau = b) + \mathbb{P}(B_\tau = -a) = 1$, we deduce that

$$\mathbb{P}(B_\tau = b) = \frac{a}{a+b}, \qquad \mathbb{P}(B_\tau = -a) = \frac{b}{a+b}.$$

Now it is clear that

$$\{B_\tau = b\} = \{T_b < T_{-a}\} \text{ and } \{B_\tau = -a\} = \{T_{-a} < T_b\}.$$

∎

Exercise 32 *For very $a, b > 0$, let us put*

$$\tau = \inf\{t \geq 0 : B_t \in]-a, b[\}.$$

Compute the expectation of τ.

The following theorem shows that any stopped martingale at a stopping time remains a martingale relative to the same filtration.

Theorem 33 *Let τ be a stopping time. Assume that $\mathbb{P}(\tau < \infty) > 0$ and put for every $t \geq 0$,*

$$B_t^{(\tau)} = \mathbf{1}_{\{\tau<\infty\}} \left(B_{\tau+t} - B_\tau\right).$$

Then under the conditional probability $\mathbb{P}(\cdot|\tau < \infty)$, the stochastic process $B^{(\tau)}$ is a Brownian motion independent of the $\sigma-$algebra $\mathcal{F}_\tau$.

Proof. The proof given here is taken from [**?**]. We begin by the case the case where $\tau > \infty$ as Let us note that we just have to prove that for every $A \in \mathcal{F}_\tau$, $0 \leq t_1 < \cdots < t_p$ and F a nonnegative bounded continuous function defined on $\mathbb{R}^p$, we have

$$\mathbb{E}(1_A F(B_{t_1}^{(\tau)}, \ldots, B_{t_p}^{(\tau)})) = \mathbb{P}(A)\mathbb{E}(F(B_{t_1}, \ldots, B_{t_p})). \tag{2.6}$$

Indeed, if $A = \Omega$ in formula (2.6) then this formula shows that $B^{(\tau)}$ is a Brownian motion. Formula (2.6) implies that for every $0 \leq t_1 < \cdots < t_p$, the random variable $(B_{t_1}^{(\tau)}, \ldots, B_{t_p}^{(\tau)})$ is independent of $\mathcal{F}_\tau$, so by a monotone class argument, it follows that $B^{(\tau)}$ is independent of $\mathcal{F}_\tau$.

For any integer $n \geq 1$, $[\tau]_n$ stands for the smallest real number of the form $k2^{-n}$ greater or equal to τ, with $[\tau]_n = \infty$ if $\tau = \infty$. In order to prove (2.6), we remark that

$$F(B_{t_1}^{(\tau)}, \ldots, B_{t_p}^{(\tau)}) = \lim_{n\to} F(B_{t_1}^{([\tau]_n)}, \ldots, B_{t_p}^{([\tau]_n)}). \tag{2.7}$$

It follows by (2.7) and dominated convergence theorem that

$$\begin{aligned}
& \mathbb{E}(\mathbf{1}_A F(B_{t_1}^{(\tau)}, \ldots, B_{t_p}^{(\tau)})) \\
= \; & \lim_{n\to\infty} \mathbb{E}(\mathbf{1}_A F(B_{t_1}^{([\tau]_n)}, \ldots, B_{t_p}^{([\tau]_n)})) \\
= \; & \lim_{n\to\infty} \sum_{k=0}^{\infty} \mathbb{E}(\mathbf{1}_A \mathbf{1}_{\{(k-1)2^{-n} < \tau \leq k2^{-n}\}} F(B_{k2^{-n}+t_1} - B_{k2^{-n}}, \ldots, B_{k2^{-n}+t_p} - B_{k2^{-n}})).
\end{aligned}$$

Now, for every $A \in \mathcal{F}_\tau$, the event $A \cap \{(k-1)2^{-n} < \tau \leq k2^{-n}\}$ is $\mathcal{F}_{k2-n}-$measurable. In view of the simple Markov property (see Propos), we have

$$\begin{aligned} & \mathbb{E}(\mathbf{1}_{A\cap\{(k-1)2^{-n}<\tau\leq k2^{-n}\}} F(B_{k2^{-n}+t_1} - B_{k2^{-n}}, \ldots, B_{k2^{-n}+t_p} - B_{k2^{-n}})) \\ = \ & \mathbb{P}(A \cap \{(k-1)2^{-n} < \tau \leq k2^{-n}\})\mathbb{E}(F(B_{t_1}, \ldots, B_{t_p})). \end{aligned}$$

It suffices to summand other k to conclude.

In the case where $\mathbb{P}(\tau = \infty) > 0$, the same arguments lead to

$$\mathbb{E}(\mathbf{1}_{A\cap\{\tau<\infty\}} F(B^{(\tau)}_{t_1}, \ldots, B^{(\tau)}_{t_p})) = \mathbb{P}(A \cap \{\tau < \infty\})\mathbb{E}(F(B_{t_1}, \ldots, B_{t_p})). \quad (2.8)$$

The conclusion follows straightly by equality (2.8). ■

A very important property of the trajectories of Brownian motion is the reflection principle which is a consequence of the Markov property we have proved above.

Theorem 34 *For every $t \geq 0$, let us put $S_t = \sup_{u\leq t} B_u$. Then, for every $a \geq 0$ and $b \leq a$, we have*

$$\mathbb{P}(S_t \geq a, B_t \leq b) = \mathbb{P}(B_t \geq 2a - b).$$

In particular, S_t and $|B_t|$ have the same probability law.

Proof. Let us note that by virtue of Theorem $T_a < \infty$ a.s. We have

$$\begin{aligned} \mathbb{P}(S_t \geq a,\ B_t \leq b) &= \mathbb{P}(T_a \leq t,\ B_t \leq b) \\ &= \mathbb{P}(T_a \leq t,\ B_t - a \leq b - a) \\ &= \mathbb{P}(T_a \leq t,\ B^{(T_a)}_{t-T_a} \leq b - a). \end{aligned}$$

Since $B^{(T_a)}$ is a Brownian motion independent of $\mathcal{F}_{T_a}$ this process is in particular independent of T_a. Therefore $(T_a, B^{(T_a)})$ and $(T_a, -B^{(T_a)})$ have the same law. Let $H = \{(s, w) \in \mathbb{R}_+ \times \mathcal{C}(\mathbb{R}_+, \mathbb{R}) : s \leq t,\ w(t-s) \leq b - a\}$. The above probability is equal to

$$\begin{aligned} \mathbb{P}((T_a, B^{(T_a)}) \in H) &= \mathbb{P}((T_a, -B^{(T_a)}) \in H) \\ &= \mathbb{P}(T_a \leq t,\ \ -B^{(T_a)}_{t-T_a} \leq b - a) \\ &= \mathbb{P}(T_a \leq t,\ B_t \geq 2a - b) \\ &= \mathbb{P}(\ B_t \geq 2a - b) \end{aligned}$$

because the $\{B_t \geq 2a - b\} \subset \{T_a \leq t\}$.

To prove that S_t and $|B_t|$ have the same probability law, we note that

$$\begin{aligned}\mathbb{P}(S_t \geq a) &= \mathbb{P}(S_t \geq a, B_t \geq a) + \mathbb{P}(S_t \geq a, B_t \leq a) \\ &= 2\mathbb{P}(B_t \geq a) \\ &= \mathbb{P}(|B_t| \geq a).\end{aligned}$$

■

Exercise 35 *Prove that* T_a *has the same law as* $\dfrac{a^2}{X}$ *where* X *follows the law* $\mathcal{N}(0,1)$. *Give the density of* T_a.

We end this chapter by giving the definition of multidimensional Brownian motion.

Definition 36 *A* $\mathbb{R}^d-$*valued process* $B = ((B_t^1, \dots, B_t^d))_{t\geq 0}$ *is called a* $d-$*dimensional Brownian motion started from* 0 *if its components* $B^1, \dots, B^d$ *are independent real Brownain motion started from* 0.

Chapter 3

Stochastic integrals

In the preceeding chapter, we have proved that a Brownian motion doesn't have finite variations on any interval. Therefore it is not possible to define an integral relative to B for fixed ω as a Stieljes integral. We will define the Itô's stochastic integral which uses a L^2 framework.

3.1 Itô's integral

Definition 37 *A stochastic process $(\phi_t)_{t\geq 0}$ is said progressively measurable if for every $t \geq 0$, the map $(\omega, s) \longmapsto \phi_t(\omega)$ defined on $\Omega \times [0,t]$ with values in $\mathbb{R}$ is $\mathcal{F}_t \otimes \mathcal{B}([0,t])-$measurable.*

Let us introduce some spaces.

$$\mathcal{M}^2(\mathbb{R}_+) = \left\{\phi : \text{ progressively measurable processes such that } \mathbb{E}\left(\int_{\mathbb{R}_+} \phi_t^2 dt\right) < \infty\right\}$$

$$\mathcal{M}^2([0,T]) = \left\{\phi : \text{ progressively measurable processes such that } \mathbb{E}\left(\int_0^T \phi_t^2 dt\right) < \infty\right\}$$

$$\mathcal{M}^2 = \bigcap_{T\geq 0} \mathcal{M}^2([0,T]).$$

It is not difficult to see that $\mathcal{M}^2(\mathbb{R}_+)$ and $\mathcal{M}^2([0,T])$ are Hilbert spaces when endowed respectively with the norms

$$\langle \phi, \psi \rangle = \int_{\mathbb{R}_+} \phi_t \psi_t dt, \quad \langle \phi, \psi \rangle = \int_0^T \phi_t \psi_t dt.$$

We begin by defining the stochastic integral for a simple process. We call simple process a real process $(\phi_t)_{t\geq 0}$ such that

$$\phi_t(\omega) = \sum_{i=0}^{n-1} X_i(\omega) 1_{]t_i,t_{i+1}]}$$

with $0 = t_0 < t_1 < t_2 < \cdots < t_n$ and X_i is a square integrable random variable which is $\mathcal{F}_{t_i}-$measurable. Let $\mathcal{E}$ stands for the set of all simple processes.

For every $\phi \in \mathcal{E}$, we define the stochstic integral of ϕ by

$$\int_{\mathbb{R}_+} \phi_t dB_t = \sum_{i=0}^{n-1} X_i[B_{t_{i+1}} - B_{t_i}].$$

It is clear that the map $\phi \longmapsto \int_{\mathbb{R}_+} \phi_t dB_t$ is linear. Let us prove that it is an isometry from $\mathcal{E}$ to $L^2(\Omega)$.

We have

$$\begin{aligned}
\mathbb{E}\left(\int_{\mathbb{R}_+} \phi_t dB_t\right)^2 &= \sum_{i=0}^{n-1} \mathbb{E}\left(X_i^2 \mathbb{E}([\mathbb{B}_{t_{i+1}} - B_{t_i}]^2 | \mathcal{F}_{t_i})\right) \\
&\quad +2\sum_{i<j} \mathbb{E}\left(X_i[\mathbb{B}_{t_{i+1}} - B_{t_i}] X_j \mathbb{E}[\mathbb{B}_{t_{j+1}} - B_{t_j}] | \mathcal{F}_{t_j}\right) \\
&= \mathbb{E}\left(\int_{\mathbb{R}_+} \phi_t^2 dt\right) = ||\phi||^2_{\mathcal{M}^2(\mathbb{R}_+)}.
\end{aligned}$$

Proposition 38 *$\mathcal{E}$ is dense in $\mathcal{M}^2(\mathbb{R}_+)$.*

Proof. See ■

Now, one can prove that the map $\mathcal{E} \ni \phi \longmapsto \int_{\mathbb{R}_+} \phi_t dB_t \in L^2(\Omega)$ admits an unique extension as an isometry from $\mathcal{M}^2(\mathbb{R}_+)$ to $L^2(\Omega)$. This extension is also denoted by $\int_{\mathbb{R}_+} \phi_t dB_t$ and called the stochstic integral of $\phi \in \mathcal{M}^2(\mathbb{R}_+)$.

The isometry property implies

Theorem 39 *For every $\phi, \psi \in \mathcal{M}^2(\mathbb{R}_+)$, we have*

$$\mathbb{E}\left(\int_{\mathbb{R}_+} \phi_t dB_t\right) = 0,\ \mathbb{E}\left(\int_{\mathbb{R}_+} \phi_t dB_t\right)^2 = \mathbb{E}\left(\int_{\mathbb{R}_+} \phi_t^2 dt\right)$$

$$\mathbb{E}\left[\left(\int_{\mathbb{R}_+} \phi_t dB_t\right)\left(\int_{\mathbb{R}_+} \psi_t dB_t\right)\right] = \mathbb{E}\left(\int_{\mathbb{R}_+} \phi_t \psi_t dt\right).$$

Remark 40 *If ϕ is a deterministic function in $\mathcal{M}^2(\mathbb{R}_+)$ that is $\phi \in L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dt)$ then $\int_{\mathbb{R}_+} \phi_t dB_t$ is the Wiener integral defined by as the element of the Gaussian space H^B with variance $\int_{\mathbb{R}_+} \phi_t^2 dt$. If G is a Gaussian measure with intensity the Lebesgue measure then $\int_{\mathbb{R}_+} \phi_t dB_t$ is also defined as $G(\phi)$. It is clear that the two definitions coincides.*

If $\phi \in \mathcal{M}^2$ then we define the stochastic integral of ϕ on $[0,t]$ by

$$\int_0^t \phi_u dB_u = \int_{\mathbb{R}_+} \mathbf{1}_{]0,t]}(u)\phi_u dB_u.$$

We put

$$\int_s^t \phi_u dB_u = \int_{\mathbb{R}_+} \mathbf{1}_{]s,t]}(u)\phi_u dB_u \ \phi_u dB_u$$

One can prove that the stochastic process $\left(\int_0^t \phi_u dB_u\right)_{t\geq 0}$ has a continuous version. From now on, we consider this continuous version.

Proposition 41 *i) Let us put $M_t = \int_0^t \phi_u dB_u$. The stochastic process M is a continuous martingale.*
ii) $\left(M_t^2 - \int_0^t \phi_u^2 du\right)_{t\geq 0}$ is a continuous martingale

Proof. Let $(t_i)_i$ be a sequence of subdivision of $[s,t]$ and $(X_i)_i$ be random variables such that X_i is $\mathcal{F}_{t_i}-$measurable and square integrable with

$$\int_s^t \phi_u dB_u = \lim \sum_i X_i[B_{t_{i+1}} - B_{t_i}], \qquad \text{in } L^2(\Omega).$$

M is $\mathcal{F}-$adapted and square integrable. Now, in $L^2(\Omega)$, we have

$$\begin{aligned} \mathbb{E}\left(\int_s^t \phi_u dB_u|\mathcal{F}_s\right) &= \lim \textstyle\sum_i \mathbb{E}(X_i[B_{t_{i+1}} - B_{t_i}]|\mathcal{F}_s) \\ &= \lim \textstyle\sum_i \mathbb{E}(X_i\mathbb{E}([B_{t_{i+1}} - B_{t_i}]|\mathcal{F}_{t_i})|\mathcal{F}_s) \\ &= 0. \end{aligned}$$

Since, $\mathbb{E}[M_t^2 - M_s^2|\mathcal{F}_s) = \mathbb{E}[(M_t - M_s)^2|\mathcal{F}_s)$, to prove that $\left(M_t^2 - \int_0^t \phi_u^2 du\right)_{t\geq 0}$ is a martingale, it is sufficient to establish that

$$\mathbb{E}[(M_t - M_s)^2|\mathcal{F}_s) = \mathbb{E}\left(\int_s^t \phi_u^2 du|\mathcal{F}_s\right).$$

We have

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_s^t \phi_u dB_u|\mathcal{F}_s\right)^2\right] &= \mathbb{E}\left(\lim_n \left[\sum_i X_i[B_{t_{i+1}} - B_{t_i}]\right]^2 |\mathcal{F}_s\right) \\
& \lim_n \left(\mathbb{E}\left[\sum_i X_i[B_{t_{i+1}} - B_{t_i}]\right]^2 |\mathcal{F}_s\right) \\
&= \lim_n \sum_i \mathbb{E}\left(\mathbb{E}\left[X_i^2[B_{t_{i+1}} - B_{t_i}]^2|\mathcal{F}_{t_i}\right]|\mathcal{F}_s\right) \\
& +2\sum_{i<j} \mathbb{E}\left(\mathbb{E}\left[X_iX_j[B_{t_{i+1}} - B_{t_i}][B_{t_{j+1}} - B_{t_j}]|\mathcal{F}_{t_j}\right]|\mathcal{F}_s\right) \\
&= \lim_n \sum_i \mathbb{E}[X_i^2(t_{i+1} - t_i)|\mathcal{F}_s] \\
&= \mathbb{E}\left(\int_s^t \phi_u^2 du|\mathcal{F}_s\right)
\end{aligned}
$$

■

3.2 Itô's formula

Let us recall the fundamental theorem of differential calculus. Let $x : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ be deterministic functions of class $\mathcal{C}^1$, Then we have

$$\Phi(x(t)) = \Phi(x(0)) + \int_0^t \Phi^{'}(x(s))x^{'}(s)ds = \Phi(x(0)) + \int_0^t \Phi^{'}(x(s))dx(s).$$

The aim of this section is to extend this formula to stochastic calculus.

Let us consider the following example. Let $t_i = it/n$ be a subdivision of $[0, t]$. We have

$$
\begin{aligned}
B_t^2 &= \sum_{i=1}^n [B_{t_i}^2 - B_{t_{i-1}}^2] \\
&= 2\sum_{i=1}^n B_{t_{i-1}}[B_{t_i} - B_{t_{i-1}}] + \sum_{i=1}^n [B_{t_i} - B_{t_{i-1}}]^2
\end{aligned}
$$

By letting $n \to \infty$, we obtain

$$B_t^2 = 2\int_0^t B_s dB_s + t.$$

Therefore the fundamental theoremm for differential calculus doesn't work in the stochastic case.

Let $\mathcal{C}_b^2$ denote the set of real valued fonctions Φ defined on $\mathbb{R}$ such that Φ and its first and second derivatives are bounded.

Theorem 42 *For every $\Phi \in \mathcal{C}_b^2$, we have a.s.*

$$\Phi(B_t) = \Phi(B_0) + \int_0^t \Phi^{'}(B_s)dB_s + \frac{1}{2}\int_0^t \Phi^{''}(B_s)ds, \qquad \forall t \geq 0$$

Proof. Let $t_i = it/n$ be a subdivision of $[0,t]$. Taylor's formula applied to Φ leads to

$$\begin{aligned} \Phi(B_t) &= \Phi(B_0) + \sum_{i=1}^n (\Phi(B_{t_i}) - \Phi(B_{t_{i-1}})) \\ &= \Phi(B_0) + \sum_{i=1}^n \Phi^{'}(B_{t_i})(\Phi(B_{t_i}) - \Phi(B_{t_{i-1}})) \\ &\quad \tfrac{1}{2}\sum \Phi^{''}(B_{\theta_i})[\Phi(B_{t_i}) - \Phi(B_{t_{i-1}})]^2, \end{aligned}$$

with $\theta_i = \theta_i(n,\omega) \in]t_{i-1}, t_i[$.

By the definition of stochastic integral, we have

$$\sum_{i=1}^n \Phi^{'}(B_{t_i})(\Phi(B_{t_i}) - \Phi(B_{t_{i-1}})) \to \int_0^t \Phi^{'}(B_s)dB_s, \text{ as } n \to \infty.$$

Now, let us put

$$U_n = \sum_{i=1}^n \Phi^{''}(B_{\theta_i})[\Phi(B_{t_i}) - \Phi(B_{t_{i-1}})]^2$$

$$V_n = \sum_{i=1}^n \Phi^{''}(B_{t_{i-1}})[\Phi(B_{t_i}) - \Phi(B_{t_{i-1}})]^2$$

and

$$W_n = \sum_{i=1}^n \Phi^{''}(B_{t_{i-1}})[t_i - t_{i-1}].$$

We have

$$\begin{aligned} \mathbb{E}|U_n - V_n| &\leq \mathbb{E}(\sup_i \left(|\Phi^{''}(B_{\theta_i}) - \Phi^{''}(B_{t_{i-1}})|\right) \times \sum_{i=1}^n [\Phi(B_{t_i}) - \Phi(B_{t_{i-1}})]^2 \\ &\leq \left[\mathbb{E}(\sup_i \left(|\Phi^{''}(B_{\theta_i}) - \Phi^{''}(B_{t_{i-1}})|^2\right) \times \left(\sum_{i=1}^n [\Phi(B_{t_i}) - \Phi(B_{t_{i-1}})]^2\right)^2\right]^{1/2}. \end{aligned}$$

Therefore by letting $n \to 0$ and thanks to Lebesgue dominated convergence Theorem and Theorem, we have $\mathbb{E}|U_n - V_n| \to 0$. We have

$$\begin{aligned}
\mathbb{E}|V_n - W_n|^2 &= \mathbb{E}\left[\left|\sum_{i=1}^n \Phi^{''}(B_{t_{i-1}})([\Phi(B_{t_i}) - \Phi(B_{t_{i-1}})]^2 - (t_i - t_{i-1}))\right|^2\right] \\
&= \sum_{i=1}^n \mathbb{E}\left(|\Phi^{''}(B_{t_{i-1}})([\Phi(B_{t_i}) - \Phi(B_{t_{i-1}})]^2 - (t_i - t_{i-1}))|^2\right) \\
&\leq \sup(\Phi^{''})^2 \sum_{i=1}^n \mathbb{E}\left(|([\Phi(B_{t_i}) - \Phi(B_{t_{i-1}})]^2 - (t_i - t_{i-1}))|^2\right) \\
&= \sup(\Phi^{''})^2 \times 2 \sum_{i=1}^n (t_i - t_{i-1})^2.
\end{aligned}$$

Therefore by letting $n \to \infty$, $\mathbb{E}|V_n - W_n|^2 \to 0$.
Since,

$$W_n = \sum_{i=1}^n \Phi^{''}(B_{t_{i-1}})[t_i - t_{i-1}] \to \int_0^t \Phi^{''}(B_s)ds,$$

we conlude. ■

Exercise 43 *For every progressively measurable process ϕ such that , $\int_0^t \phi_s^2 ds < \infty$, $t \geq 0$ we put*

$$Z_t = \exp\left(\int_0^t \phi_s dB_s - \frac{1}{2}\int_0^t \phi_s^2 ds\right).$$

Prove that the stochastic process Z is a martingale.

Exercise 44 *Let $X = (X_t)_{t\geq 0}$ be an Itô process that is*

$$X_t = X_0 + \int_0^t \phi_s dB_s + \int_0^t \psi_s ds,$$

where $\phi, \psi \in \mathcal{M}^2$ and X_0 is a square integrable and $\mathcal{F}_0-$measurable random variable. Prove that for every $\Phi \in \mathcal{C}_b^2$

$$\Phi(X_t) = \Phi(X_0) + \int_0^t \Phi^{'}(X_s)\phi_s dB_s + \int_0^t \Phi^{'}(X_s)\psi_s ds + \frac{1}{2}\int_0^t \Phi^{''}(X_s)\phi_s^2 ds. \tag{3.1}$$

Formula (3.1) may be write in differential form

$$dX_t = \Phi^{'}(X_t)dX_t + \frac{1}{2}\Phi^{''}(X_s)d\langle X\rangle_t$$

where

$$\langle X\rangle_t = \int_0^t \phi_s^2 ds.$$