The theory of A_p weights: a modern introduction[1]

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1 The Hardy-Littlewood maximal function.

The Hardy-Littlewood maximal function is the (non-linear) operator defined by

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|\,dy$$

where the supremum is taken over all the cubes containing x. By cube we always mean a cube with sides parallel to the axes. There are variations of this definition such as replacing cubes by balls or just considering cubes or balls centered at x but all of them are equivalent except for dimensional constants. (See [Jo] for more information about all these).

Maximal functions arise very naturally in analysis, for proving theorems about the existence almost everywhere of limits, for controlling pointwise important objects such as the Poisson Integrals or for controlling, not pointwise but at least in average, other basic operators such as singular integral operators.

The model example of existence almost everywhere of limits is the **Lebesgue differentiation theorem:**

$$f(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy \tag{1}$$

and this property is intimately related to the study of certain properties of the Hardy-Littlewood maximal function. There are other almost everywhere convergence problems in mathematics: Fourier series, Dirichlet problem, the heat equation, the Schrodinger equation etc. All of them have the same pattern, study first the associated maximal operator.

The key property to understand the Lebesgue differentiation theorem is the so called "weak type" estimate or property of M. This is related to a

[1]Notes for the CIMPA ABIDJAN 2014 school.

very special an fundamental space, the weak spaces or Marcinkiewicz spaces:

$$\|f\|_{L^{p,\infty}(\mathbb{R}^n)} = \sup_{t\geq 0} t\,|\{x \ \in \mathbb{R}^n : |f(x)| > t\}|^{1/p} \qquad p > 0.$$

It is easy to see that $L^{p,\infty}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ and that the content is strict.

In applications it is specially important the case $p = 1$.

Theorem 1.1 (Hardy–Littlewood–Wiener) *M is a weak type $(1,1)$ operator, namely*

$$M : L^{1,\infty}(\mathbb{R}^n) \to L^1(\mathbb{R}^n),$$

which means that there exists a finite (dimensional) constant c such that the following inequality holds,

$$\|Mf\|_{L^{1,\infty}(\mathbb{R}^n)} \leq c\,\|f\|_{L^1(\mathbb{R}^n)} \tag{2}$$

Corollary 1.2 *M is of strong type (p,p), $1 < p < \infty$ operator, namely, there exists a constant c such that*

$$\|Mf\|_{L^p(\mathbb{R}^n)} \leq c\|f\|_{L^p(\mathbb{R}^n)} \tag{3}$$

Sometimes we will be using the notation

$$\|T\|_{L^{p,\infty}(X)} = \sup_{f\neq 0} \frac{\|Tf\|_{L^{p,\infty}(X)}}{\|f\|_{L^p(X)}}$$

or the standard one

$$\|T\|_{L^p(X)} = \sup_{f\neq 0} \frac{\|Tf\|_{L^p(X)}}{\|f\|_{L^p(X)}}$$

As a consequence of (2) we can derive the Lebesgue differentiation theorem (1). To be more precise we will show that there is a set L_f the **Lebesgue set** whose complement N_f is a "null" set, namely $|N_f| = 0$ for which

$$f(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\,dy \qquad x \in L_f \tag{4}$$

Observe that this is equivalent to saying

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)|\,dy = 0 \qquad x \in L_f$$

To do this we define

$$A_\lambda = \{x : \limsup_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)|\, dy > \lambda\} \qquad \lambda \geq 0$$

It is enough to prove that $|A_\lambda| = 0$ for each $\lambda > 0$ since $A_0 = \cup_j A_{1/j}$ and since $x \notin A_0$, implies

$$0 = \limsup_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)|\, dy$$

but then we can replace the lim sup by lim, namely,

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)|\, dy = 0$$

which is exactly (4) if we choose $N_f = A_0$.

We now prove then that $|A_\lambda| = 0$ for each $\lambda > 0$. A first important remark is that (4) holds a priori for a dense class of functions, for instance for class of continuous functions in $\mathbb{R}^n$:

$$g(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} g(y)\, dy \qquad x \in \mathbb{R}^n$$

Now, if $f \in L^1(\mathbb{R}^n)$ and for given $\epsilon > 0$ there exists a continuous function g such that $\|f - g\|_{L(\mathbb{R}^n)} < \varepsilon$. Then g satisfies

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |g(y) - g(x)|\, dy = 0 \qquad x \in \mathbb{R}^n.$$

So, if $x \in A_\lambda$ and write $f = g + h$ we immediately have

$$\lambda < \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)|\, dy \leq Mh(x) + |h(x)|$$

and then

$$|A_\lambda| \leq |\{x \in \mathbb{R}^n : Mh(x) > \lambda/2\}| + |\{x \in \mathbb{R}^n : h(x) > \lambda/2\}|$$

and here comes the main point to apply (2):

$$|A_\lambda| \leq \frac{c_n}{\lambda} \int_{\mathbb{R}^n} |h(x)|\, dx = \frac{c_n}{\lambda} \|f - g\|_{L(\mathbb{R}^n)} < \frac{c_n}{\lambda} \varepsilon$$

But ε is free so we can let $\varepsilon \to 0$ obtaining $|A_\lambda| = 0$.

The moral: a crude estimate such as a weak type implies a very saddle result, the Lebesgue differentiation theorem which involves cancellation of the function f around x because the function change sign.

2 A first but remarkable weighted estimate

In this section we will prove Theorem 1.1 and its Corollary 1.2, actually we will prove a weighted version for the same effort. This weighted estimates are very important due to its many applications and because they open the theory of weights.

We say that w is a weight if w is a a.e. positive locally integrable function in $\mathbb{R}^n$. If E is any measurable set we denote $w(E) = \int_E w$. Sometimes we can beyond locally integrable functions as is the case of the main theorem below.

Theorem 2.1 *a) There exists a constant c_n such that for all f, w*

$$\|Mf\|_{L^{1,\infty}(w)} \leq c_n \int_{\mathbb{R}^n} |f(x)|\, Mw(x)dx. \tag{5}$$

b) As a consequence if $1 < p < \infty$ there exists a constant C such that for all f

$$\|Mf\|_{L^p(w)} \leq c_n\, (p')^{1/p}\, \|f\|_{L^p(Mw)} \tag{6}$$

We may think of (5) inequality as a kind of duality for the (nonlinear) operator. It can also be seen as an antecedent of the A_p theory weights that we will be developing.

One important tool is the so called "the layer cake formula" formula will plays a central role:

$$\int_X \varphi(f)\, d\mu = \int_0^\infty \varphi'(t)\, \mu(\{x \in X : f(x) > t\})\, dt \qquad f \geq 0 \tag{7}$$

Proof of Theorem 2.1:

We will beging with the proof of (6). We will use part a), namely the weak type estimate(5). We may assume that f is nonnegative. Write f as $f = f^t + f_t$, where $f^1(x) = f(x)$ if $f(x) > t$, and $f^t(x) = 0$ otherwise. Then

$$Mf(x) \leq Mf^t(x) + Mf_t(x) \leq Mf^t(x) + t.$$

Hence, by the cake layer formula

$$\int_{\mathbb{R}^n} Mf(y)^p\, w(y)dy = p \int_0^\infty t^p w\{y \in \mathbb{R}^n : Mf(y) > t\}\, \frac{dt}{t} = p2^p \int_0^\infty t^p w\{y \in \mathbb{R}^n : Mf(y) > 2t\}$$

$$\leq p2^p \int_0^\infty t^p w\{y \in \mathbb{R}^n : Mf^t(x) > t\} \frac{dt}{t} \leq c_n\, p2^p \int_0^\infty t^{p-1} \int_{\{y \in \mathbb{R}^n : f(y) > t\}} f(y)\, Mw(y)dy\, \frac{dt}{t}$$

by the key weak type inequality (5). Now we continue with

$$\leq c_n\, p\, 2^p \int_{\mathbb{R}^n} \int_0^{f(y)} t^{p-1} \frac{dt}{t} f(y)\, Mw(y)dy = C\, p'\, 2^p \int_{\mathbb{R}^n} f(y)^p\, Mw(y)dy$$

and this concludes the proof of (3)

A first proof of (5)**:** The proof of a) is based on the classical Vitali covering Lemma.

Lemma 2.2 (Vitali) *Let $\mathcal{F} = \{Q_i\}_{i=1,\cdots,N}$ be a finite family of cubes (or balls) in $\mathbb{R}^n$. Then we can extract from $\mathcal{F}$ a sequence of pairwise* **disjoint** *cubes $\mathcal{F}' = \{Q_j\}_{j=1,\cdots,M}$ such that*

$$\bigcup_{Q \in \mathcal{F}} Q \subset \bigcup_{j=1}^{m} 5Q_j.$$

For the proof we remit [Ma] p.23 but there are many others good references. The key point is the a geometrical point: the so called "engulfing" property of balls.

If we let $\Omega_\lambda = \{x \in \mathbb{R}^n : Mf(x) > \lambda\}$ for a given λ and let K to be any compact subset contained in Ω_λ. Let $x \in K$ then by defintion of the maximal function there is a cube $Q = Q_x$ containg x such that

$$\frac{1}{|Q|} \int_Q |f(y)|\, dy > \lambda. \tag{8}$$

Then $K \subset \bigcup_{x \in K} Q_x$ and by compactness we can extract a finite family of cubes $\mathcal{F} = \{Q\}$ such that $K \subset \bigcup_{Q \in \mathcal{F}} Q$ and where each cube satisfies (8). Then by the Vitali lemma we can extract a pairwise disjoint family of cubes $\{Q_j\}_{j=1}^M$ such that $\bigcup_{Q \in \mathcal{F}} \subset \bigcup_{j=1}^M 5Q_j$. Then

$$w(K) \leq \sum_{j=1}^M w(5Q_j) \leq \frac{1}{\lambda} \sum_{j=1}^M w(5Q_j) \frac{1}{|Q_j|} \int_{Q_j} |f(y)|\, dy$$

$$\leq \frac{C}{\lambda} \sum_{j=1}^M \frac{w(5Q_j)}{|5Q_j|} \int_{Q_j} |f(y)|\, dy \leq \frac{C}{\lambda} \sum_{j=1}^M \int_{Q_j} |f(y)|\, Mw(y)dy$$

$$\leq \frac{C}{\lambda}\int_{\mathbb{R}^n} |f(y)|\, Mw(y)dy.$$

A second proof of (5)**:** This second proof is just an application of the Besicovitch covering lemma which plays a central role in modern analysis.

Lemma 2.3 (The Besicovitch covering lemma) *Let K de a bounded set in $\mathbb{R}^n$ and suppose that for every $x \in K$ there is an (open) cube $Q(x)$ with center at x. Then we can find a sequence (possible finite) of points $\{x_j\}$ in K such that*

$$K \subset \bigcup_j Q(x_j)$$

and

$$\sum_j \chi_{Q(x_j)} \leq c_n$$

where c_n is a finite dimensional constant. In fact we can write

$$\bigcup_j Q(x_j) = \bigcup_{k=1}^{B(n)} \bigcup_{i\in\mathcal{F}_k} Q_i, \tag{9}$$

where each of the family $\{Q_i\}_{i\in\mathcal{F}_k}$, $k = 1, \cdots, B(n)$, is formed by pairwise disjoint cubes. $B(n) > 1$ is usually called the Besicovitch constant.

The proof of this result can be found in several places such as the classical lecture notes by M. De Guzman [dGuz] [Ma], also in [GrMF].

Indeed, assuming that w is bounded as we may, the first observation is that (5) is equivalent to

$$\sup_{\lambda>0} \lambda\, w(\{x \in \mathbb{R}^n : M(f\frac{w}{Mw})(x) > \lambda\}) \leq C \int_{\mathbb{R}^n} |f(x)|\, w(x)dx \qquad \lambda > 0.$$

The second is that we trivially have the pointwise inequality

$$M(f\frac{w}{Mw})(x) \leq c_n\, M_w^c f(x),$$

where M_w^c is the weighted centered maximal function

$$M_w^c f(x) = \sup_{r>0} \frac{1}{w(Q_r(x))} \int_{Q_r(x)} |f(y)|\, w(y)dy. \tag{10}$$

Therefore (5) follows from

$$\sup_{\lambda>0} \lambda\, w(\{x \in \mathbb{R}^n : M_w^c f(x) > \lambda\}) \leq C \int_{\mathbb{R}^n} |f(x)|\, w(x)dx$$

which is a consequence of the Besicovitch covering lemma where c si a dimensional constant.

□

The proof of b) is a particular case of one of the most important theorems in modern mathematics: the **Marcinkiewicz interpolation theorem**. An extension of above argument making a special emphasis on the control on the bound is the following estimate:

$$\|T\|_{L^p(X)} \leq 2\,(p')^{1/p} \left(\|T\|_{L^{1,\infty}(X)}\right)^{1/p} \left(\|T\|_{L^\infty(X)}\right)^{1/p'}.$$

2.1 An application: a vector-valued inequality

Inequality (5) was proved by C. Fefferman and E. Stein in a celebrated paper [FS] to derive a vector–valued extension of (3). Indeed, for a vector of functions $f = \{f_i\}_i$ and for $q > 0$ the operator $\overline{M}_q$ is defined by

$$\overline{M}_q f(x) = \left(\sum_{i=1}^{\infty} (Mf_i(x))^q\right)^{1/q}.$$

This operator can also be seen as a generalization of the classical integral of Marcinkiewicz. Then if we denote $|f(x)|_q = (\sum_{i=1}^{\infty} |f_i(x)|^q)^{1/q} = \|f(x)\|_{\ell^q}$ we have for $1 < p < \infty$ and $1 < q \leq \infty$

$$\int_{\mathbb{R}^n} \overline{M}_q f(x)^p\, dx \leq C \int_{\mathbb{R}^n} |f(x)|_q^p\, dx. \tag{11}$$

A corresponding weak type $(1,1)$ also holds in the range $1 < p < \infty$. These estimates play an important role in Harmonic Analysis, specially in the theory of Littlewood-Paley.

If $p = q$ there is nothing to be proved, the case $p > q$ and let $r = \frac{p}{q} > 1$. By the Fatout lemma we may assume that the series is finite from $i = 1$ to $i = N$ and then let $N \to \infty$. Raise (11) to the power $\frac{1}{r}$ we can write

$$\left\|\sum_{i=1}^{N} (Mf_i)^q\right\|_{L^r(\mathbb{R}^n)} = \sum_{i=0}^{N} \int_{\mathbb{R}^n} Mf_i(y)^q\, g(y)dy$$

for some $g \in L^{r'}(\mathbb{R}^n)$ with unit norm. But then we can apply (6) to continue with

$$\leq c_n q' \sum_{i=0}^{N} \int_{\mathbb{R}^n} f_i(y)^q \, Mg(y)dy, \leq c_n q' \int_{\mathbb{R}^n} \sum_{i=0}^{N} f_i(y)^q \, Mg(y)dy$$

$$\leq c_n q' \| \sum_{i=0}^{N} f_i(y)^q \|_{L^r(\mathbb{R}^n)} \|Mg\|_{L^{r'}(\mathbb{R}^n)}$$

and by (6) (with no weight) we continue with

$$\leq c_n q' r^{1/r'} \left(\int_{\mathbb{R}^n} |f(x)|_q^p \, dx \right)^{1/r} \|g\|_{L^{r'}(\mathbb{R}^n)} = c_n q' r^{1/r'} \left(\int_{\mathbb{R}^n} |f(x)|_q^p \, dx \right)^{1/r}$$

concluding the proof.

The case $p \leq q$ is easy but follows from vector-valued interpolation and can be found in [GCRdF]

2.2 The A_1 condition

The question is a one weighted question, namely for which weights w we have the boundedness

$$M : L^1(w) \to L^{1,\infty}(w)$$

In view of the Fefferman-Stein's inequality (5) these authors defined the following class of weights.

Definition 2.4 *A weight w belongs to the class A_1 if there is a constant K such that*

$$\frac{1}{|Q|} \int_Q w(y) \, dy \leq K \inf_Q w, \tag{12}$$

and we will denote the infimum of the constant K by $[w]_{A_1}$.

Observe that $K \geq 1$ by the Lebesgue differentiation theorem. And with this notation we have

$$\|M\|_{L^{1,\infty}(w)} \leq c_n [w]_{A_1}.$$

So this means that the A_1 condition is a sufficient condition for the boundedness $M : L^1(w) \to L^{1,\infty}(w)$ but this condition is also necessary. In fact we will show that

$$\|M\|_{L^{1,\infty}(w)} \geq [w]_{A_1}$$

in an easy way in next section (see (15)).

It is important to have large class of A_1 weights to our disposal and there are eentially two methods. The first method is due to Coifman-Rochberg [CR].

Lemma 2.5 *Let μ be a positive Borel, then for each* $0 < \delta < 1$

$$(M\mu)^\delta \in A_1$$

and furthermore

$$[(Mf)^\delta]_{A_1} \leq \frac{c_n}{1-\delta}. \tag{13}$$

The second method is based in the so called Rubio de Francia's algorithm

Lemma 2.6 *For any measurable function f define If*

$$R(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{M^k(h)}{\|M\|^k_{L^p(w)}}$$

Then we have

(A) $h \leq D(h)$

(B) $\|R(h)\|_{L^p(w)} \leq 2\,\|h\|_{L^p(w)}$

(C) $[R(h)]_{A_1} \leq 2\,\|M\|_{L^p(w)}$

More information about this can be found in [CMP3].

3 Muckenhoupt A_p condition and Muckenhoupt A_p theorem

The main breakthrough in the theory arrived in the early seventies with the discovery by B. Muckenhoupt in [Mu] of the right class of weights for which the Hardy-Littlewood maximal function is bounded on $L^p(w)$, $p > 1$. This result opened up the possibility of studying operators with higher singularity. We show here non standard proofs

Definition 3.1 *A weight w belongs to the class A_p, $1 < p < \infty$, if*

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|}\int_Q w(y)\,dy\right)\left(\frac{1}{|Q|}\int_Q w(y)^{1-p'}\,dy\right)^{p-1} < \infty \tag{14}$$

Check that by Jensen's inequality

$$1 \le p \le q \quad \Longrightarrow \quad [w]_{A_q} \le [w]_{A_p} \quad \text{i.e.} \quad A_p \subset A_q$$

We know prove Muckenhoupt‘s theorem with the sharp constant due to S. Buckley. It came as a surprised that this theorem can be improved as shown in [HP] an further simplified in [HPR]. To state these new results we introduce the concept of A_∞ constant. This recent results are intimately related to the so called A_2 conjecture and its improvement.

Theorem 3.2 *Let $p > 1$. The following conditions are equivalent:*

a) $$w \in A_p;$$

b) there exists a constant K such that for each cube Q and nonnegative function f

$$\left(\frac{1}{|Q|}\int_Q f(y)\,dy\right)^p w(Q) \le [w]_{A_p}\int_Q f(y)^p\, w(y)dy$$

c) $$M : L^p(w) \to L^{p,\infty}(w) \text{ and furthermore}$$

$$[w]_{A_p}^{\frac{1}{p}} \le \|M\|_{L^{p,\infty}(w)} \le c_n\,[w]_{A_p}^{\frac{1}{p}}. \tag{15}$$

d) $$M : L^p(w) \to L^p(w) \text{ and furthermore}$$

$$\|M\|_{L^p(w)} \le c_n\,\left(p'[w]_{A_p}\right)^{\frac{1}{p-1}}, \tag{16}$$

where $1 < p < \infty$. Furthermore the exponent is sharp in the sense that: for any $\theta > 0$

$$\sup_{w\in A_p} \frac{1}{[w]_{A_p}^{\frac{1}{p-1}-\theta}}\|M\|_{L^p(w)} = \infty \tag{17}$$

A new way to understand the sharpeness of the exponent in (16) can be found in [LPR]. It turns out that it is due to the fact that the norm of M as operator on $L^p(\mathbb{R}^n)$ blows up as $\frac{1}{p-1}$ when $p \to 1$.

As already mentioned estimate (16) has been improved quite substantially in [HP] and will be derived in Theorem 5.7.

In fact, we cannot replace the function $\psi(t) = t^{\frac{1}{p-1}}$ by a "smaller" function $\psi : [1, \infty) \to (0, \infty)$ in the sense that

$$\inf_{t>1} \frac{\psi(t)}{t^{\frac{1}{p-1}}} = 0$$

(or $\lim_{t\to\infty} \frac{\psi(t)}{t^\beta} = 0$, or $\sup_{t>1} \frac{t^\beta}{\psi(t)} = \infty$ or $\lim_{t\to\infty} \frac{t^\beta}{\psi(t)} = \infty$) since then

$$\sup_{w\in A_p} \frac{1}{\psi([w]_{A_p})} \|M\| = \infty \tag{18}$$

Conditions a), b) and c) are also equivalent when $p = 1$. These and d) are essentially due to Muckenhoupt except for the sharp constant. We use a different approach. The sharp constant in d) is due to Buckley [Bu] however we follow here the approach by Lerner [L]. Later on we will see a further improvement.

Proof:

Since $w \in A_p$ for each cube Q and nonnegative function f, by Holders inequality

$$\left(\frac{1}{|Q|}\int_Q f\,dy\right)^p w(Q) = \left(\frac{1}{|Q|}\int_Q f w^{1/p}\, w^{-1/p} dy\right)^p w(Q) \le [w]_{A_p} \int_Q f(y)^p\, w(y)dy.$$

Hence

$$Mf(x) \le 2^n M^c f(x) \le 2^n\, [w]_{A_p}^{\frac{1}{p}}\, M_w^c(f^p)(x)^{\frac{1}{p}},$$

and by the Besicovitchs lemma:

$$\|Mf\|_{L^{p,\infty}(w)} \le c_n [w]_{A_p}^{\frac{1}{p}}\, \|M_w^c(f^p)^{\frac{1}{p}}\|_{L^{p,\infty}(w)} = c_n [w]_{A_p}^{\frac{1}{p}}\, \|M_w^c(f^p)\|_{L^{1,\infty}(w)}^{\frac{1}{p}}$$

$$\le c_n [w]_{A_p}^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} f^p\, w\, dx\right)^{1/p}$$

This yieds that a) implies b) implies c).

To prove d) Since this theorem is so central we give another proof. we set

$$A_p(Q) = \frac{w(Q)}{|Q|}\Big(\frac{\sigma(3Q)}{|Q|}\Big)^{p-1}$$

then, we have using that

$$A_p(Q) \leq 3^{np}\,[w]_{A_p}$$

and that for any $x \in Q$ then $Q \subset Q(x, 2\ell(Q)) \subset 3Q$

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f| &= A_p(Q)^{\frac{1}{p-1}}\left\{\frac{|Q|}{w(Q)}\Big(\frac{1}{\sigma(3Q)}\int_Q |f|\Big)^{p-1}\right\}^{\frac{1}{p-1}} \\ &\leq 3^{np'}\,[w]_{A_p}^{\frac{1}{p-1}}\left\{\frac{1}{w(Q)}\int_Q M_\sigma^c(f\sigma^{-1})^{p-1}dx\right\}^{\frac{1}{p-1}}.\end{aligned}$$

where M_σ^c is the weighted centered maximal function. Using again that

$$Mf(x) \leq 2^n M^c f(x)$$

we get

$$Mf(x) \leq\, 2^n\,3^{np}\,[w]_{A_p}^{\frac{1}{p-1}}\,\left\{M_w^c\big(M_\sigma^c(f\sigma^{-1})^{p-1}w^{-1}\big)(x)\right\}^{\frac{1}{p-1}}$$

We conclude using that both

$$\|M_w^c\|_{L_w^{p'}} \qquad \text{and} \qquad \|M_\sigma^c\|_{L_\sigma^p}$$

are finite with constants uniformly in w. This follows from the Besicovitch covering Lemma.

Another way: by dyadic, passing from dyadic to non dyadic by "shifting".

For the sharpness we consider $n = 1$ and $0 < \varepsilon < 1$. Let

$$w(x) = |x|^{(1-\varepsilon)(p-1)}.$$

It is easy to check that

$$[w]_{A_p}^{\frac{1}{p-1}} \approx \frac{1}{\varepsilon}$$

and hence as in Buckley's paper

$$f(y) = y^{-1+\varepsilon}\,\chi_{(0,1)}(y)$$

Observe that:

$$\|f\|^p_{L^p(w)} \approx \frac{1}{\varepsilon}$$

To estimate now $\|Mf\|_{L^p(w)}$ we pick $0 < x < 1$, hence

$$Mf(x) \geq \frac{1}{x}\int_0^x f(y)\,dy = \frac{1}{\varepsilon}f(x)$$

and hence

$$\|Mf\|_{L^p(w)} \geq \frac{1}{\varepsilon}\,\|f\|_{L^p(w)}$$

From which the rest follows easily.

$\square$

4 A factorization theorem with control on the constants

Muckenhoupt already observed in [M] that it follows from the definition of the A_1 weights that if $w_1, w_2 \in A_1$, then

$$w = w_1 w_2^{1-p} \in A_p,$$

and furthermore

$$[w]_{A_p} \leq [w_1]_{A_1}[w_2]^{p-1}_{A_1} \tag{19}$$

In view of this Muckenhoupt conjectured that any A_p weight can be written in this way. This theorem was P. Jones who proved the necessity of the factorization theorem with a very difficult proof. The modern approach to this theorem is based on completely different arguments and it is due to J. L. Rubio de Francia. Here we give a proof of this result using sharp constants. We will use a variation of the method of Rubio de Francia which appears in [H].

It is also well known that the modern approach to this question uses completely different path and it is due to J. L. Rubio de Francia as can be found in [GrMF] where we remit the reader for more information about the A_p theory of weights. Here we present a variation which appears in [H]. Here we give a proof of this result using sharp constants. To be more precise we have the following result.

Lemma 4.1 *Let* $1 < p < \infty$ *and let* $w \in A_p$, *then there are* A_1 *weights* $u, v \in A_1$, *such that*

$$w = u\, v^{1-p}$$

in such a way that

$$[u]_{A_1} \le c_n [w]_{A_p} \quad \& \quad [v]_{A_1} \le c_n [w]_{A_p}^{\frac{1}{p-1}} \tag{20}$$

Proof :

Following [H] we will be using the Rubio de Francia's iteration scheme or algorithm to our situation. Define

$$S_1(f)^{p'} \equiv w^{1/p}\, M(\frac{|f|^{p'}}{w^{1/p}}),$$

and

$$S_2(f)^p \equiv \frac{1}{w^{1/p}}\, M(|f|^p w^{1/p}),$$

Observe that $S_i : L^{pp'}(\mathbb{R}^n) \to L^{pp'}(\mathbb{R}^n)$ with constant

$$\|S_i\|_{L^{pp'}(\mathbb{R}^n)} \le c_n [w]_{A_p}^{1/p} \qquad i = 1, 2$$

by Buckley's theorem.

Now, the operator $S = S_1 + S_2$ is bounded on $L^{pp'}(\mathbb{R}^n)$ with

$$\|S\|_{L^{pp'}(\mathbb{R}^n)} \le c_n [w]_{A_p}^{1/p}.$$

Define the Rubio de Francia algorithm R by

$$R(h) \equiv \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{S^k(h)}{(\|S\|_{L^{pp'}(\mathbb{R}^n)})^k}.$$

Observe that R is also bounded on $L^{pp'}(\mathbb{R}^n)$. Now, if $h \in L^{pp'}(\mathbb{R}^n)$ is fixed, $R(h) \in A_1(S)$. More precisely

$$S(R(h)) \le 2\, \|S\|_{L^{pp'}(\mathbb{R}^n)} \le c_n [w]_{A_p}^{1/p}.$$

In particular $R(h) \in A_1(S_i)$ $i = 1, 2$, with

$$S_i(R(h)) \le c_n [w]_{A_p}^{1/p} R(h) \qquad i = 1, 2$$

Hence

$$M(R(h)^{p'}w^{-1/p}) \le c_n[w]_{A_p}^{p'/p} R(h)^{p'}w^{-1/p}$$

and

$$M(R(h)^{p}w^{1/p}) \le c_n[w]_{A_p} R(h)^{p}w^{1/p}.$$

Finally, letting

$$u \equiv R(h)^p w^{1/p} \qquad \& \qquad v \equiv R(h)^{p'} w^{-1/p}$$

we have $u, v \in A_1$ $w = uv^{1-p}$ with

$$[u]_{A_1} \le c_n[w]_{A_p} \quad \& \quad [v]_{A_1} \le c_n[w]_{A_p}^{\frac{1}{p-1}}$$

□

5 Improving the A_p theorem via the A_∞ constant

We know that the class of weights is increasing and hence it is natural to define the A_∞ class of weights as follows.

Definition 5.1 *The A_∞ class of weights is defined in a natural way by*

$$A_\infty := \cup_{p>1} A_p.$$

The A_∞ class of weights shares a lot of interesting properties.

It is known that an equivalent way of defining the class A_∞ is by means of a the following quantity

$$[w]_{A_\infty}^{expL} := \sup_Q \fint_Q w \, \exp\Big(\fint_Q -\log w\Big).$$

This constant can be found in for instance in [GCRdF] and was introduced by Hruščev in [Hr]. This definition is natural because all you have to do is to let $p \to \infty$ in the deifniition of the A_p constant. In fact we have by Jensen's inequality:

$$[w]_{A_\infty}^{expL} \le [w]_{A_p}.$$

On the other hand in [HP] the authors use a "new" A_∞ constant (which was essentially introduced by Fujii in [Fujii] and rediscovered later by Wilson in [Wil87]) which is more suitable.

Definition 5.2

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q)\ dx.$$

Observe that $[w]_{A_p} \geq 1$ by the Lebesgue differentiation theorem.

It is shown in [HP] that

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q)\ dx. \tag{21}$$

In fact, it is shown in [HP] that there are examples showing that $[w]_{A_\infty}$ is much smaller than $[w]^{expL}_{A_\infty}$.

There is another possible way of defying the A_∞ class of weights. Indeed, define

$$[w]^{LlogL}_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q w(y) \log(e + \frac{w(y)}{w_Q})\, dy$$

since because it can be shown that

$$[w]^{LlogL}_{A_\infty} \leq 2^{n+1}\, [w]_{A_\infty}$$

which follows from the following lemma.

Lemma 5.3 *For any cube Q and any measurable function w,*

$$\int_Q w\ \log(e + \frac{w}{w_Q})\, \mathrm{d}x \leq 2^{n+1} \int_Q M(w\chi_Q)\, \mathrm{d}x, \tag{22}$$

and hence if $w \in A_\infty$

$$[w]^{LlogL}_{A_\infty} \leq 2^{n+1}\, [w]_{A_\infty}$$

Inequality (22) plays an important role in many situations in particular when dealing with local estimates. The essential idea behind the proof can be traced back to the well known $L \log L$ estimate for M in [St] (see also [Wil-LNM, p. 17, inequality (2.15)] for a different proof). A proof is supplied in [HP].

5.1 The Reverse Hölder property

In the classical situation, if $w \in A_\infty$, then there is a constant $r > 1$ such that

$$\left(\frac{1}{|Q|}\int_Q w^r\right)^{1/r} \le \frac{c}{|Q|}\int_Q w$$

Observe that the other inequality is immediate by Jensen's inequality. This property has played a central role in the area. The standard proofs show that there is a **bad** dependence on the constant $c = c(r, [w]_{A_1})$. To improve our results we need a more precise estimate.

The main result is the following

Theorem 5.4 (Sharp Reverse Hölder Inequality)
a) Let $w \in A_\infty$ and let Q_0 be a cube. Then

$$\left(\fint_Q w^{r(w)}\right)^{1/r(w)} \le 2\fint_Q w \le 2\fint_{Q_0} w\, dx,$$

where $r(w) = \frac{1}{c_d\,[w]_{A_\infty}}$.

b) Furthermore, the result is optimal up to a dimensional factor: If a weight w satisfies the reverse Hölder inequality

$$\left(\fint_Q w^r\right)^{1/r} \le K\fint_Q w,$$

then $[w]_{A_\infty} \le c_n \cdot K \cdot r'$.

This result was obtained in [HP] (using Sawyer'e discretization method) and has been improved in [HPR] with a much simpler proof. A previous reverse Holder's inequality in this spirit was obtained in [LOP1] but for A_1 weights.

Part b) follows from the boundedness of the maximal function in L^r with constant $c_n r'$:

$$\begin{aligned}\fint_Q M(\chi_Q w) &\le \left(\fint_Q M(\chi_Q w)^r\right)^{1/r} \\ &\le c_n \cdot r'\left(\fint_Q w^r\right)^{1/r} \le c_n \cdot r' \cdot K\fint_Q w.\end{aligned}$$

In this section we give a different proof of Theorem 5.4. The constant is a slightly worst but the method is more direct and may be of interest. The main idea is to linearize the maximal function following Sawyer's well know method but instead at global level at local one.

The main use of the Lemma is the following key observation that we borrow from [Wil-LNM], p. 45:

Lemma 5.5 *Let* $S \subset Q$ *and let* $\lambda > 0$*, then*

$$\frac{|S|}{|Q|} < e^{-\lambda} \quad \textit{implies} \quad \frac{w(S)}{w(Q)} < \frac{2^{d+2}[w]_{A_\infty}}{\lambda} + e^{-\lambda/2} \tag{23}$$

proof: Indeed, if $E_\lambda = \{x \in Q : w(x) > e^\lambda \langle w \rangle_Q\}$ then $w(E_\lambda) \leq \frac{2^{d+1}}{\lambda}[w]_{A_\infty} w(Q)$ by (22). Therefore:

$$\begin{aligned} w(S) &\leq w(S \cap E_{\lambda/2}) + w(S \setminus E_{\lambda/2}) \leq \frac{2^{d+2}\,[w]_{A_\infty}}{\lambda}\, w(Q) + e^{\lambda/2} \langle w \rangle_Q \, |S| \\ &\leq \frac{2^{d+2}\,[w]_{A_\infty}}{\lambda}\, w(Q) + e^{\lambda/2} e^{-\lambda}\, w(Q) \qquad \text{by the hypothesis in (23)} \\ &= \frac{2^{d+2}\,[w]_{A_\infty}}{\lambda}\, w(Q) + e^{-\lambda/2}\, w(Q) \end{aligned}$$

and this proves the claim (23).

□

proof [of Theorem 5.4] Recall that we have to prove that

$$\left(\fint_Q w^{r(w)} \right)^{1/r(w)} \leq 2 \fint_Q w.$$

where

$$r(w) := 1 + \frac{1}{\tau_d\, [w]'_{A_\infty}},$$

and where τ_d is a large dimensional constant.

Observe that by homogeneity we can assume that $\fint_Q w = 1$. We use the dyadic maximal function on the dyadic subcubes of a given Q:

$$\begin{aligned}
\int_Q w^{1+\epsilon} &\leq \int_Q M_d(w\chi_Q)^\epsilon w = \int_0^\infty \epsilon t^{\epsilon-1} w(\{x\in Q : M_d(w\chi_Q)>t\})\,\mathrm{d}t. \\
&\leq \int_0^1 \epsilon t^{\epsilon-1} w(Q)\,\mathrm{d}t + \epsilon \int_1^\infty \epsilon t^{\epsilon} w(\{x\in Q : M_d(w\chi_Q)>t\})\frac{dt}{t} \\
&\leq |Q| + \epsilon \sum_{k\geq 0} \int_{a^k}^{a^{k+1}} t^{\epsilon} w(\{x\in Q : M_d(w\chi_Q)>t\})\frac{dt}{t} \\
&\leq |Q| + \epsilon a^{\epsilon} \sum_{k\geq 0} a^{k\epsilon} \int_{a^k}^{a^{k+1}} w(\{x\in Q : M_d(w\chi_Q)>a^k\})\frac{dt}{t}, \quad \text{for } a \gg 1, \\
&= |Q| + \epsilon a^{\epsilon}\, \log a \sum_{k\geq 0} a^{k\epsilon}\, w(\Omega_k)
\end{aligned}$$

where

$$\Omega_k = \{x\in Q : M_d(w\chi_Q(x) > a^k\}.$$

Since $a^k \geq 1 = \fint_Q w$ we can consider the Calderón–Zygmund decomposition w adapted to Q. There is a family of maximal non-overlapping dyadic cubes $\{Q_{k,j}\}$ strictly contained in Q for which $\Omega_k = \bigcup_j Q_{k,j}$ and

$$a^k < \fint_{Q_{k,j}} w \leq 2^d a^k. \tag{24}$$

Now,

$$\sum_{k\geq 0} a^{k\epsilon}\, w(\Omega_k) = \sum_{k,j} a^{k\epsilon} w(Q_{k,j}) \leq \sum_{k,j} \left(\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} w(y)\,dy\right)^{\epsilon} w(Q_{k,j})$$

We now estimate $w(Q_{k,j})$, for each (k,j) we set $E_{k,j} = Q_{k,j}\setminus \Omega_{k+1}$. Observe that the sets of the family $E_{k,j}$ are pairwise disjoint. We claim that for $a > 2^n$ and for each k, j:

$$|Q_{k,j}| < \frac{a}{a-2^n}\,|E_{k,j}|. \tag{25}$$

We now apply (23) with $Q = Q_{k,j}$ and $S = Q_{k,j}\cap\Omega_{k+1}$. Choose λ such that $e^{-\lambda} = \frac{2^n}{a}$, namely $\lambda = \log\frac{a}{2^n}$. Then applying (23) we have that

$$\frac{w(Q_{k,j}\cap\Omega_{k+1})}{w(Q_{k,j})} < \frac{2^{d+2}\,[w]_{A_\infty}}{\log\frac{a}{2^n}} + (\frac{2^n}{a})^{1/2}.$$

Since $a > 2^n$ is available we choose $a = 2^d e^{L\,[w]_{A_\infty}}$, with L a large dimensional constant to be chosen. If in particular $L \geq 2^{d+4}$ we have

$$\frac{w(Q_{k,j} \cap \Omega_{k+1})}{w(Q_{k,j})} < \frac{2^{d+2}}{L} + e^{-\,[w]_{A_\infty} L/2} < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

This yields that $w(Q_{k,j}) \leq 2w(E_{k,j})$ and we can continue with the sum estimate:

$$\sum_{k\geq 0} a^{k\epsilon}\, w(\Omega_k) \leq 2 \sum_{k,j} \left(\frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} w(y)\,\mathrm{d}y \right)^{\epsilon} w(E_{k,j})$$

$$\leq 2 \sum_{k,j} \int_{E_{k,j}} M_d(w\chi_Q)^{\epsilon}\, w dx \leq 2 \int_Q M_d(w\chi_Q)^{\epsilon}\, w dx$$

Combining estimates we end up with

$$\fint_Q M_d(w\chi_Q)^{\epsilon} w \leq 1 + 2\,\epsilon a^{\epsilon}\, \log a \fint_Q M_d(w\chi_Q)^{\epsilon}\, w dx$$

for any $\epsilon > 0$. Recall that $a = 2^d e^{L\,[w]_{A_\infty}}$. Hence if we choose

$$L = 2^{d+4}, \qquad \epsilon = \frac{1}{2^7 L\,[w]_{A_\infty}} = \frac{1}{2^{11+d}\,[w]_{A_\infty}},$$

we can compute

$$2\,\epsilon a^{\epsilon}\, \log a < \frac{1}{2}, \qquad \fint_Q M_d(w\chi_Q)^{\epsilon} w \leq 2,$$

concluding the proof of the theorem.

□

5.2 The open property

As a corollary we deduce the following useful result.

Corollary 5.6 (The Precise Open property) *Let* $1 < p < \infty$ *and let* $w \in A_p$*. Then* $w \in A_{p-\epsilon}$ *where*

$$\epsilon = \frac{p-1}{r'_\sigma} = \frac{p-1}{1+\tau_n\,[\sigma]_{A_\infty}}$$

where as usual $\sigma = w^{1-p'}$*. Furthermore*

$$[w]_{A_{p-\varepsilon}} \leq 2^{p-1}\,[w]_{A_p}$$

Proof
Since $w \in A_p$ $\sigma \in A_{p'} \subset A_\infty$, and hence

$$⨍_Q w\Big(⨍_Q \sigma^{r_\sigma}\Big)^{\frac{p-1}{r_\sigma}} \leq ⨍_Q w\Big(2⨍_Q \sigma\Big)^{p-1}.$$

Choose ε so that $\frac{p-1}{r_\sigma} = p-\varepsilon-1$, namely $\epsilon = \frac{p-1}{r'_\sigma}$. Observe that $\varepsilon > 0$ and $p-\varepsilon > 1$.

$\square$

5.3 Improving Muckenhoupt-Buckley's theorem

Now we prove one of the main result.

Theorem 5.7 *Let* $p > 1$.

$$\|M\|_{L^p(w)} \leq c_n\,\left(p'[w]_{A_p}[\sigma]_{A_\infty}\right)^{1/p}, \tag{26}$$

where $1 < p < \infty$, $\sigma = w^{\frac{1}{1-p}}$.

Proof: We will use the weak type estimate (15) and

$$\{x \in \mathbb{R}^n : Mf(x) > 2t\} \subset \{x \in \mathbb{R}^n : Mf_t(x) > t\}$$

where $f_t = f\,\chi_{f>t}$. Now, since $w \in A_p$ by "The Precise Open property" Corollary 5.6 if $w \in A_{p-\epsilon}$, where $\epsilon = \frac{p-1}{r(\sigma)'}$ and $[w]_{A_{p-\varepsilon}} \leq 2^{p-1}\,[w]_{A_p}$.
Then

$$\int_{\mathbb{R}^n} (Mf)^p\,w dx = p\int_0^\infty t^p w\{y \in \mathbb{R}^n : Mf(x) > t\}\,\frac{dt}{t} = p2^p\int_0^\infty t^p w\{y \in \mathbb{R}^n : Mf(x) > 2t\}\,\frac{dt}{t}$$

$$\leq p\, 2^p \int_0^\infty t^p w\{y \in \mathbb{R}^n : M f_t(x) > t\} \,\frac{dt}{t} \leq p\, c_n^{p-\varepsilon}\, [w]_{A_{p-\varepsilon}}\, 2^p \int_0^\infty t^p \int_{\mathbb{R}^n} \frac{f_t(y)^{p-\varepsilon}}{t^{p-\varepsilon}}\, w(y)dy\, \frac{dt}{t}$$

$$\leq p\, c_n^{p-\varepsilon}\, 2^{p-1}\, [w]_{A_p}\, 2^p \int_{\mathbb{R}^n} \int_0^{f(y)} t^\varepsilon\, \frac{dt}{t} f(y)^{p-\varepsilon}\, w(y)dy = \frac{1}{\varepsilon} 2^{2p-1}\, p\, c_n^{p-\varepsilon}\, [w]_{A_p} \int_{\mathbb{R}^n} f(y)^p\, w(y)dy$$

$$= \frac{r(\sigma)'}{p-1}\, p\, 2^{2p-1}\, c_n^{p-\varepsilon}\, [w]_p \int_{\mathbb{R}^n} f(y)^p\, w(y)dy = \frac{1+\tau_n[\sigma]_{A_\infty}}{p-1}\, p\, 2^{2p-1}\, c_n^{p-\varepsilon}\, [w]_p \int_{\mathbb{R}^n} f(y)^p\, w(y)dy$$

and this yields (26) .

□

Of course we recover here part d) of Theorem (3.2):

$$\|M\|_{L^p(w)} \leq cp'\, [w]_{A_p}^{\frac{1}{p-1}}$$

since

$$[\sigma]_{A_\infty} \leq [\sigma]_{A_{p'}} = [w]_{A_p}^{\frac{1}{p-1}}$$

6 The sharp extrapolation theorem

This result was proved in [DGPP] but it is found [CMP3] a much simpler proof. (see also [D]).

Theorem 6.1 *Let $1 < p_0 < \infty$, and $\alpha > 0$ and let T be any operator such that for some $c > 0$*

$$\|T\|_{L^{p_0}(w)} \leq c\, [w]_{A_{p_0}}^{\alpha}. \tag{27}$$

Then, for $1 < p < p_0$ there is universal $c > 0$ such that

$$\|T\|_{L^p(w)} \leq c\, [w]_{A_p}^{\alpha \max\{1, \frac{p_0-1}{p-1}\}}. \tag{28}$$

Proof. Let $1 < p < p_0$. For such a p and $w \in A_p$ perform the iteration algorithm R as follows:

$$R(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{M^k(h)}{\|M\|_{L^p(w)}^k}$$

Then we have

(A) $h \leq D(h)$

(B) $\|R(h)\|_{L^p(w)} \leq 2\,\|h\|_{L^p(w)}$

(C) $[R(h)]_{A_1} \leq 2\,\|M\|_{L^p(w)} \leq c_n\, p'\; [w]_{A_p}^{\frac{1}{p-1}}$
by Theorem 3.2, d).
Then

$$\|T(f)\|_{L^p(w)} = \Big(\int_{\mathbb{R}^n} |Tf|^p\,(Rf)^{-(p_0-p)\frac{p}{p_0}}\,(Rf)^{(p_0-p)\frac{p}{p_0}}\,wdx\Big)^{1/p}$$

$$\leq \Big(\int_{\mathbb{R}^n} |Tf|^{p_0}\,(Rf)^{-(p_0-p)}\,wdx\Big)^{1/p_0}\Big(\int_{\mathbb{R}^n}(Rf)^p\,wdx\Big)^{\frac{p_0-p}{pp_0}}$$

$$\leq c\,[R(f)^{-(p_0-p)}\,w]_{A_{p_0}}^{\alpha}\Big(\int_{\mathbb{R}^n}|f|^{p_0}\,(Rf)^{-(p_0-p)}\,wdx\Big)^{1/p_0}\|f\|_{L^p(w)}^{\frac{p_0-p}{p_0}}$$

$$\leq c\,[R(f)^{-(p_0-p)}\,w]_{A_{p_0}}^{\alpha}\Big(\int_{\mathbb{R}^n}|f|^{p}\,wdx\Big)^{1/p_0}\|f\|_{L^p(w)}^{\frac{p_0-p}{p_0}}$$

$$= c\,[R(f)^{-(p_0-p)}\,w]_{A_{p_0}}^{\alpha}\,\|f\|_{L^p(w)}$$

To conclude we claim the following:

$$[R(f)^{-(p_0-p)}\,w]_{A_{p_0}} \leq [w]_{A_p}^{\frac{p_0-1}{p-1}}$$

Indeed for any cube and the definition of A_1

$$\frac{1}{|Q|}\int_Q R(f)^{-(p_0-p)}\,wdx \leq\; [Rf]_{A_1}^{p_0-p}\Big(\frac{1}{|Q|}\int_Q R(f)\,dx\Big)^{-(p_0-p)}\,\frac{1}{|Q|}\int_Q wdx$$

and also if we set $q = \frac{p_0-1}{p_0-p} > 1$ and observe that $q' = \frac{p_0-1}{p-1}$. Then, Holder's gives

$$\frac{1}{|Q|}\int_Q\Big(R(f)^{-(p_0-p)}\,w\Big)^{1-p_0'}\,dx = \frac{1}{|Q|}\int_Q R(f)^{\frac{p_0-p}{p_0-1}}\,w^{1-p_0'}\,dx$$

$$\leq \Big(\frac{1}{|Q|}\int_Q R(f)\,dx\Big)^{\frac{p_0-p}{p_0-1}}\Big(\frac{1}{|Q|}\int_Q w^{1-p'}\,dx\Big)^{\frac{p-1}{p_0-1}}$$

Hence, combining we have

$$[R(f)^{-(p_0-p)}\, w]_{A_{p_0}} \leq c\, [Rf]_{A_1}^{p_0-p}\, [w]_{A_p} \leq c\, \|M\|_{L^p(w)}^{p_0-p}\, [w]_{A_p} \leq [w]_{A_p}^{\frac{p_0-p}{p-1}}\, [w]_{A_p} = [w]_{A_p}^{\frac{p_0-1}{p-1}}$$

Let $p > p_0$. For such a p and $w \in A_p$ perform the iteration algorithm R' as follows:

$$R'(h) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{(M')^k(h)}{\|M'\|_{L^p(w)}^k}$$

where M' is the operator defined by $\frac{M'(fw)}{w}$. Observe that by Theorem 3.2 part d) $\|M'\|_{L^p(w)} = \|M\|_{L^p(\sigma)} \leq c_n\, p'\, [\sigma]_{A_{p'}}^{\frac{1}{p'-1}} = c_n\, p'\, [w]_{A_p}$. Then it is easy to verify that

(A') $h \leq R'(h)$

(B') $\|R'(h)\|_{L^{p'}(w)} \leq 2\, \|h\|_{L^{p'}(w)}$

(C') $[R'(h)w]_{A_1} \leq 2\, \|M'\|_{L^{p'}(w)} \leq c_n\, p\, [w]_{A_p}$ Now by duality,

$$\|T(f)\|_{L^p(w)} = \sup_h \int |Tf| hw dx$$

for any non-negative $h \in L^{p'}(w)$ such that $\|f\|_{L^p(w)} = 1$. Now we set $W = (R'(h))^{\frac{p-p_0}{p-1}} w$ and claim that $W \in A_{p_0}$. Assuming the claim for the time being we use Holder's inequality together with the extrapolation hypothesis to derive

$$\int_{\mathbb{R}^n} |Tf|^p\, (R'h)^{\frac{p-p_0}{p_0(p-1)}}\, (R'h)^{\frac{(p_0-1)p}{p_0(p-1)}}\, wdx$$

$$\leq \Big(\int_{\mathbb{R}^n} |Tf|^{p_0}\, Wdx \Big)^{1/p_0} \Big(\int_{\mathbb{R}^n} (R'h)^{p'}\, wdx \Big)^{\frac{1}{p_0}}$$

$$\leq c_{p_0,T,} [W]_{A_{p_0}}^{\alpha} \Big(\int_{\mathbb{R}^n} |f|^{p_0}\, (R'h)^{\frac{p-p_0}{p-1}}\, dx \Big)^{1/p_0}$$

$$\leq c_{p_0,T,}\, [W]_{A_{p_0}}^{\alpha} \Big(\int_{\mathbb{R}^n} |f|^p\, wdx \Big)^{1/p}\, |R'(h)\|_{L^{p'}(w)}^{\frac{p-p_0}{p_0(p-1)}}$$

$$\leq c_{p_0,T,}\, [W]_{A_{p_0}}^{\alpha} \Big(\int_{\mathbb{R}^n} |f|^p\, wdx \Big)^{1/p}.$$

To conclude we claim the following:

$$[W]_{A_{p_0}} \leq c_{n,p,p_0}\, [w]_{A_p}$$

We now estimate $[W]_{A_{p_0}}$: Indeed by Holder's inequality with exponent $q = \frac{p-1}{p-p_0} = \frac{(p/p_0)'}{p'} > 1$ and $q' = \frac{p-1}{p_0-1}$.

$$\frac{1}{|Q|}\int_Q W dx = \frac{1}{|Q|}\int_Q (R'(h))^{\frac{p-p_0}{p-1}} w dx \le \Big(\frac{1}{|Q|}\int_Q R'(h)\, w dx\Big)^{1/q}\Big(\frac{1}{|Q|}\int_Q w dx\Big)^{\frac{p_0-1}{p-1}}$$

but on the other hand by the A_1 property from (C')

$$\Big(\frac{1}{|Q|}\int_Q W^{1-p_0'} dx\Big)^{p_0-1} = \Big(\frac{1}{|Q|}\int_Q (R'(h))^{-\frac{p-p_0}{(p-1)(p_0-1)}} w^{1-p_0'}\Big)^{p_0-1}$$

$$\le [R'(h)w]_{A_1}^{\frac{p-p_0}{p-1}}\Big(\frac{1}{|Q|}\int_Q R'(h)w\Big)^{-\frac{p-p_0}{p-1}}\Big(\frac{1}{|Q|}\int_Q w^{\frac{p-p_0}{(p-1)(p_0-1)}} w^{1-p_0'} dx\Big)^{p_0-1}$$

$$= [R'(h)w]_{A_1}^{\frac{p-p_0}{p-1}}\Big(\frac{1}{|Q|}\int_Q R'(h)w\Big)^{-\frac{p-p_0}{p-1}}\Big(\frac{1}{|Q|}\int_Q w^{1-p'} dx\Big)^{p_0-1}$$

Combining estimates we have and using again condition (C')

$$\frac{1}{|Q|}\int_Q W dx\Big(\frac{1}{|Q|}\int_Q W^{1-p_0'} dx\Big)^{p_0-1} \le [R'(h)w]_{A_1}^{\frac{p-p_0}{p-1}}\Big(\frac{1}{|Q|}\int_Q w dx\Big)^{\frac{p_0-1}{p-1}}\Big(\frac{1}{|Q|}\int_Q w^{1-p'} dx\Big)^{p_0-1}$$

$$\le [R'(h)w]_{A_1}^{\frac{p-p_0}{p-1}}\, [w]_{A_p}^{\frac{p_0-1}{p-1}} \le c\, [w]_{A_p}.$$

□

Applications to Singular Integrals, commutators, square functions, vector-valued maximal function and other operators can be found in [Hy], [HL], [HLP], [HP], [CMP1], [CMP2], [ChPP].

7 Two weight problem: sharp Sawyer's theorem

Recall Sawyer's theorem [S]:

Let $1 < p < \infty$, then there is a finite C

$$\|M(f\sigma)\|_{L^p(u)} \le C\, \|f\|_{L^p(\sigma)} \qquad (29)$$

if and only if there is a finite constant K such that for any cube Q

$$\left(\int_Q |M(\sigma\,\chi_Q)|^p u\,dx\right)^{1/p} \leq K\,\sigma(Q)^{1/p}$$

7.1 Moen version

Kabe Moen version of Sawyer's theorem: let

$$[u,\sigma]_{S_p} = \sup_Q \frac{\left(\int_Q |M(\sigma\,\chi_Q)|^p u\,dx\right)^{1/p}}{\sigma(Q)^{1/p}}$$

Theorem 7.1 *[M] Let $1 < p < \infty$ and let $\|M\|$ the smallest of the constants C in (29). Then*

$$[u,\sigma]_{S_p} \leq \|M\| \leq c_n\,p'\,[u,\sigma]_{S_p}$$

Very recently, with E. Rela we have found an application of this result and as a particular case we obtain the following ([PR]).

Corollary 7.2

$$\|M\| \leq c_n\,p'\,([u,\sigma]_{A_p}[\sigma]_{A_\infty})^{1/p} \tag{30}$$

Proof (skecth): Let $a > 1$ a universal parameter to be chosen. Fix a cube Q and let k_0 an integer such that $a^{k_0} \leq \fint_Q \sigma < a^{k_0+1}$. Then

$$\int_Q |M(\sigma\,\chi_Q)|^p u\,dx = \int_{M(\sigma\,\chi_Q)\leq a^{k_0}} |M(\sigma\,\chi_Q)|^p u\,dx + \sum_{k=k_0}^{\infty} \int_{a^k<M(\sigma\,\chi_Q)\leq a^{k+1}} |M(\sigma\,\chi_Q)|^p u\,dx$$

$$\leq (\fint_Q \sigma)^p u(Q) + a^p \sum_{k=k_0}^{\infty} a^{kp} u(\{x\in Q : M(\sigma\,\chi_Q) > a^k\})$$

but by the standard local Calderón–Zygmund covering lemma since $a^k > \fint_Q \sigma$, $k \geq k_0$ we can find dyadic relative with respect to Q subcubes $\{Q_{k,j}\}$ such that for each of these k

$$\{x\in Q : M(\sigma\,\chi_Q) > a^k\} = \cup_{k,j} Q_j^k$$

and

$$a^k < \fint_{Q_j^k} \sigma \le 2^n a^k \qquad j \in \mathbb{Z}$$

Furthermore the family of cubes satisfies the sparness property. Then using the A_p condition we continue with t

$$\le [u,\sigma]_{A_p}\sigma(Q)+a^p \sum_{k,j} \left(\fint_{Q_j^k} \sigma\, dx\right)^p u(Q_j^k) \le [u,\sigma]_{A_p}\sigma(Q)+a^p \sum_{k,j} \left(\fint_{Q_j^k} \sigma\, dx\right)^{p-1} \fint_{Q_j^k} u\, dx\, \sigma(Q_j^k)$$

$$\le [u,\sigma]_{A_p}\sigma(Q) + c_n\, [u,\sigma]_{A_p} \sum_{j,k} \sigma(Q_j^k)$$

Now by the sparness property of the cubes we can continue last sum with

$$\le c_n \sum_{j,k} \fint_{Q_j^k} \sigma\, dx\, |E_j^k| \le c_n \sum_{j,k} \int_{E_j^k} M(\chi_Q \sigma)\, dx \le c_n \int_Q M(\chi_Q\sigma)\, dx \le c_n\, [\sigma]_{A_\infty}\, \sigma(Q)$$

since the family E_j^k is pairwise disjoint and all of them contained in Q. Putting together we are done.

$\square$

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