Intrinsic square functions on functions spaces including weighted Morrey spaces

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Outline

1. Introduction
2. Norms inequalities for intrinsic square function
3. Norm inequalities for intrinsic Littlewood-Paley g^*_λ-function
4. Commutator of those operators

Wiener amalgam spaces

Let $0 < p, q \leq \infty$. For $r > 0$, put :

$$
{}_r\|f\|_{q,p} = \begin{cases} \left(\int_{\mathbb{R}^n} \left(|B(y,r)|^{-\frac{1}{p}} \left\|f\chi_{B(y,r)}\right\|_q\right)^p dy\right)^{\frac{1}{p}} & \text{if} \quad p < \infty \\ \operatorname{ess\,sup}_{y\in\mathbb{R}^n} \left\|f\chi_{B(y,r)}\right\|_q & \text{if} \quad p = \infty \end{cases},
$$

and

$$
(L^q, L^p) = \left\{ f : \mathbb{R}^n \to \mathbb{C} \text{ measurable} : \|f\|_{q,p} := {}_1\|f\|_{q,p} < \infty \right\}
$$

- $L^q \cup L^p \subseteq (L^q, L^p)$ if $q \leq p$
- $(L^q, L^p) \subseteq L^q \cap L^p$ if $p \leq q$

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Some dilation invariant subspaces of (L^q, L^p)

For $\alpha > 0$ and $r > 0$, we define $\delta_r^\alpha : f \mapsto r^{\frac{d}{\alpha}} f(r\cdot)$.

Definition

For $1 \leq q, p, \alpha \leq \infty$,

$$(L^q, L^p)^\alpha = \left\{ f \in (L^q, L^p) : \|f\|_{q,p,\alpha} := \sup_{r>0} \|\delta_r^\alpha f\|_{q,p} < \infty \right\}$$

$$\|f\|_{q,p,\alpha} \approx \begin{cases} \sup\limits_{r>0} \left[\int_{\mathbb{R}^d} \left(|B(y,r)|^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\| f\chi_{B(y,r)} \right\|_q dy \right)^p \right]^{\frac{1}{p}} & \text{if} \quad p < \infty \\ \sup\limits_{r>0} \operatorname*{sup\,ess}\limits_{y\in\mathbb{R}^d} |B(y,r)|^{\frac{1}{\alpha}-\frac{1}{q}} \left\| f\chi_{B(y,r)} \right\|_q & \text{if} \quad p = \infty \end{cases}$$

$(L^q, L^p)^\alpha = L^\alpha$ if $\alpha \in \{p, q\}$, $L^\alpha \hookrightarrow (L^q, L^p)^\alpha \hookrightarrow (L^q, L^p)$

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Norm inequalities of some classical operators

- Maximal operator defined by

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|\, dy$$

are bounded on $(L^q, L^p)^\alpha$

- The Hilbert operator ($n = 1$) is bounded on $(L^q, L^p)^\alpha$.
- Some CZO are bounded on $(L^q, L^p)^\alpha$
- For $< \gamma < n$, the Riesz potential $I_\gamma f(x) \approx \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}} dy$is bounded from $(L^q, L^p)^\alpha$ to $(L^{\widetilde{q}}, L^{\widetilde{p}})^{\alpha^*}$ for $1 < q \leq \alpha \leq p \leq \infty$ and $0 < \frac{\gamma}{d} < \frac{1}{\alpha}$, where $\frac{1}{\alpha^*} = \frac{1}{\alpha} - \frac{\gamma}{d}$. , $\frac{1}{\widetilde{q}} = \frac{1}{q} - \frac{\alpha}{q}\frac{\gamma}{d}$ and $\frac{1}{\widetilde{p}} = \frac{1}{p} - \frac{\alpha}{p}\frac{\gamma}{d}$.

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Weighted $(L^q, L^p)^\alpha$ spaces

Let $1 \leq q \leq \alpha \leq p \leq \infty$ and w be a weight. For $f : \mathbb{R}^n \to \mathbb{C}$,

$\|f\|_{q_w} := \left(\int_{\mathbb{R}^n} |f(x)|^q w(x)dx\right)^{\frac{1}{q}}, \quad w(B) = \int_B w(x)dx$

${}_r\|f\|_{q_w,p,\alpha} := \left[\int_{\mathbb{R}^n} \left(w(B(y,r))^{\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p}} \left\|f\chi_{B(y,r)}\right\|_{q_w}\right)^p dy\right]^{\frac{1}{p}}$

$\|f\|_{q_w,p,\alpha} := \sup_{r>0} {}_r\|f\|_{q_w,p,\alpha}$

and the usual modification when $p = \infty$.

Definition

$(L^q_w, L^p)^\alpha(\mathbb{R}^n) = \left\{f : \mathbb{R}^n \to \mathbb{C} \text{ measurable } : \|f\|_{q_w,p,\alpha} < \infty\right\}$

1. For $w \equiv 1$, we obtain $(L^q, L^p)^\alpha(\mathbb{R}^n)$
2. For $q < \alpha$ and $p = \infty$, $(L^q_w, L^\infty)^\alpha(\mathbb{R}^n)$ is the weighted Morrey spaces $L^{q,\kappa}_w(\mathbb{R}^n)$, with $\kappa = \frac{1}{q} - \frac{1}{\alpha}$

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Some maximal function

Let $0 < \gamma \leq 1$, $\mathbb{B} = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $\varphi : \mathbb{R}^n \to \mathbb{C}$.

1. The function $\varphi \in \mathcal{C}_\gamma$ iff
   - $\mathrm{supp}\varphi \subset \mathbb{B}$
   - $\int_{\mathbb{R}^n} \varphi(x)dx = 0$
   - $|\varphi(x) - \varphi(x')| \leq |x - x'|^\gamma, \quad \forall x, x' \in \mathbb{R}^n$
2. For $f \in L^1_{loc}$ put

$$f^*_\gamma(y,t) = \sup_{\varphi \in \mathcal{C}_\gamma} |f * \varphi_t(y)|, (y,t) \in \mathbb{R}^{n+1}_+,$$

where $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$ and $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$.

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Intrinsic square function

Definition

We defined S_γ by

$$S_\gamma(f)(x) = \left[\int_{\Gamma(x)} f_\gamma^*(y,t)^2 \frac{dydt}{t^{n+1}}\right]^{\frac{1}{2}},$$

where for $x \in \mathbb{R}^n$, $\Gamma(x) = \left\{(y,t) \in \mathbb{R}_+^{n+1} : |x-y| < t\right\}$.

Norm inequality for $\mathcal{S}_\gamma$

Theorem [Wilson]

Let $w \in \mathcal{A}_q$, $1 < q < \infty$. Then

$$\left\| S_\gamma f \right\|_{q_w} \preccurlyeq \left\| f \right\|_{q_w} .$$

Theorem [Wang]

Let $1 < q < \infty$, $0 < \kappa < 1$ and $w \in \mathcal{A}_q$. Then

$$\left\| S_\gamma f \right\|_{L_w^{q,\kappa}} \preccurlyeq \left\| f \right\|_{L_w^{q,\kappa}} .$$

Theorem [F.]

Let $0 < \gamma \leq 1$, $1 < q \leq \alpha < p \leq \infty$ and $w \in \mathcal{A}_q$. The operators S_γ are bounded in $(L_w^q, L^p)^\alpha(\mathbb{R}^n)$,i.e.,

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Intermediate result

Proposition [F.]

Let $1 \leq s \leq q < \infty$, $w \in \mathcal{A}_{q/s}$ and $\mathcal{T} : L^q_{\text{loc}}(w) \to L^q_{\text{loc}}(w)$ a sub linear operator which satisfies the following property : for all balls $B \subset \mathbb{R}^n$

$$\mathcal{T}(f\chi_{(2B)^c})(x) \preccurlyeq \sum_{k=1}^{\infty} k \left(\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |f(z)|^s \, dz \right)^{\frac{1}{s}} \text{ a.e. on } B.$$

Then

1. if $q > 1$ and $\mathcal{T}$ is bounded on $L^q(w)$, then it is also bounded on $(L^q(w), L^p)^\alpha$, for $q \leq \alpha < p \leq \infty$,
2. if for all $\lambda > 0$

$$w(\{x \in \mathbb{R}^n : |\mathcal{T}f(x)| > \lambda\} \leq C \frac{1}{\lambda} \int_{\mathbb{R}^n} |f(y)| \, w(y) dy,$$

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Sketch of the proof

Fix $y \in \mathbb{R}^n$ and $r > 0$. For a.e. $x \in B = B(y, r)$,

$$|\mathcal{T}f(x)| \preccurlyeq |\mathcal{T}(f\chi_{2B})(x)| + \sum_{k=1}^{\infty} \frac{k}{w(2^{k+1}B)^{\frac{1}{q}}} \|f\chi_{2^{k+1}B}\|_{L^q(w)}$$

- $q > 1$ take the $L^q(w)$-norm on B

$$\begin{aligned}\|\mathcal{T}f\chi_B\|_{L^q(w)} &\preccurlyeq \|f\chi_{2B}\|_{L^q(w)} \\ &+ \sum_{k=1}^{\infty} k \|f\chi_{2^{k+1}B}\|_{L^q(w)} \left(\frac{w(B)}{w(2^{k+1}B)}\right)^{\frac{1}{q}} .\end{aligned}$$

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$$\preccurlyeq \|f\chi_{2B}\|_{L^1(w)} + \sum_{k=1}^{\infty} \frac{kw(B)}{w(2^{k+1}B)} \|f\chi_{2^{k+1}B})\|_{L^1(w)}$$

Proof of the theorem

Fix $B = B(y, r)$ a ball and let $f_2 = f\chi_{(2B)^c}$. For $t > 0$, we have

$$|f_2 * \varphi_t(u)| \preccurlyeq t^{-n} \int_{(2B)^c \cap \tilde{B}(u,t)} |f(z)|\, dz, \varphi \in \mathcal{C}_\gamma, u \in \mathbb{R}^n.$$

Let $x \in B$, we have

$$|S_\gamma(f_2)(x)| \preccurlyeq \left[\int_{\Gamma(x)} \left(t^{-n} \int_{(2B)^c \cap \tilde{B}(u,t)} |f(z)|\, dz\right)^2 \frac{dudt}{t^{n+1}}\right]^{\frac{1}{2}}$$

$$\overset{\text{Minkowski}}{\preccurlyeq} \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |f(z)| \left[\int_0^\infty \left(\int_{B(x,t)} \chi_{\tilde{B}(z,t)}(u) du\right) \frac{dt}{t^{3n+1}}\right]^{\frac{1}{2}} dz$$

Proof of the theorem

$$
\begin{aligned}
|S_\gamma(f_2)(x)| &\preccurlyeq \sum_{k=1}^{\infty} \int_{2^{k+1}B\backslash 2^k B} |f(z)| \left(\int_{2^{k-2}r}^{\infty} \int_{B(x,t)} du \frac{dt}{t^{3n+1}} \right)^{\frac{1}{2}} dz \\
&\preccurlyeq \sum_{k=1}^{\infty} \int_{2^{k+1}B\backslash 2^k B} |f(z)| \left(\int_{2^{k-2}r}^{\infty} \frac{dt}{t^{2n+1}} \right)^{\frac{1}{2}} dz \\
&\preccurlyeq \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B\backslash 2^k B} |f(z)| \, dz.
\end{aligned}
$$

Outline

Intrinsic Littlewood-Paley g_λ^*-function

g_λ^*-function $g_{\lambda,\gamma}^*(f)$ is defined by

$$g_{\lambda,\gamma}^*(f)(x) = \left[\int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} f_\gamma^*(y,t)^2 \frac{dydt}{t^{n+1}}\right]^{\frac{1}{2}},$$

A norm control of Litlewood-Paley intrinsic operator

Theorem [Theorem 1.3, Wang]

Let $0 < \gamma \leq 1$, $1 < q < \infty$, $0 < \kappa < 1$ and $w \in \mathcal{A}_q$. If $\lambda > \max\{q, 3\}$, then

$$\left\| g^*_{\lambda,\gamma} f \right\|_{L^{q,\kappa}_w} \preccurlyeq \|f\|_{L^{q,\kappa}_w} .$$

Theorem [F.]

Let $0 < \gamma \leq 1$, $1 < q \leq \alpha < p \leq \infty$ and $w \in \mathcal{A}_q$. If $\lambda > \max\{q, 3\}$ then

$$\left\| g^*_{\lambda,\gamma}(f) \right\|_{q_w,p,\alpha} \preccurlyeq \|f\|_{q_w,p,\alpha}, f \in (L^q_w, L^p)^\alpha(\mathbb{R}^n)$$

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Relation between the two operators

Let $0 < \gamma \leq 1$ and $\beta > 0$.

- Define $S_{\gamma,\beta}(f)$ by

$$S_{\gamma,\beta}(f)(x) = \left[\int_{\Gamma_\beta(x)} f_\gamma^*(y,t)^2 \frac{dydt}{t^{n+1}}\right]^{\frac{1}{2}},$$

where $\Gamma_\beta(x) = \left\{(x,t) \in \mathbb{R}_+^{n+1} / \, |x-y| < \beta t\right\}$.

- We have

$$g_{\lambda,\gamma}^*(f)(x)^2 \preccurlyeq S_\gamma(f)(x)^2 + \sum_{j=1}^{\infty} 2^{-j\lambda n} S_{\gamma,2^j}(f)(x)^2.$$

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Boundedness on weighted Lebesgue space

Lemma [Wang]

Let $0 < \gamma \leq 1$, $1 < q < \infty$ and $w \in \mathcal{A}_q$. Then for all non negative integers j, $S_{\gamma,2^j}$ is bounded on $L^q_w(\mathbb{R}^n)$. Moreover

$$\left\|S_{\gamma,2^j}(f)\right\|_{q_w} \preccurlyeq (2^{nj} + 2^{\frac{njq}{2}}) \left\|f\right\|_{q_w} .$$

Sketch of the proof

The same arguments we use to estimate $S_\gamma(f_2)(x)$ for $x \in B$, i.e., Minkowsky's integral inequality and the fact that for $k \in \mathbb{N}^*$, $z \in 2^{k+1}B \setminus 2^k B$

$$\int_{B(x,2^j t)} \chi_{\tilde{B}(z,t)}(u)du \neq 0 \Rightarrow t \geq \frac{2^{k-1}}{2^j+1} r,$$

allow us to get the following

$$\left|S_{\gamma,2^j}(f_2)(x)\right| \preccurlyeq 2^{3jn/2} \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B \setminus 2^k B} |f(z)|\, dz$$

for all $x \in B(y, r)$.

Outline

Commutator of $\mathcal{S}_{\gamma,\lambda}$ and $g^*_{\lambda,\gamma}$

Let $b \in L^1_{\text{loc}}$. The commutator $[b, S_\gamma]$ is defined by

$$[b, S_\gamma](f)(x) = \left(\int_{\Gamma(x)} [(b(x) - b)f]^*_\gamma (y,t)^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}},$$

and $\left[b, g^*_{\lambda,\gamma}\right]$ by

$$\left[b, g^*_{\lambda,\gamma}\right](f)(x) = \left\{ \int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t + |x-y|} \right)^{\lambda n} [(b(x) - b)f]^*_\gamma (y,t)^2 \frac{dydt}{t^{n+1}} \right\}^{\frac{1}{2}}$$

Norm inequalities

Theorem [Wang]

Let $0 < \gamma \leq 1$, $1 < q < \infty$ and $w \in \mathcal{A}_q$. Then $[b, S_\gamma]$ and $\left[b, g^*_{\lambda,\gamma}\right]$ are bounded on $L^q_w(\mathbb{R}^n)$ whenever $b \in BMO(\mathbb{R}^n)$.

Theorem [F.]

Let $0 < \gamma \leq 1$, $1 < q \leq \alpha < p \leq \infty$ and $w \in \mathcal{A}_q$. Suppose that $b \in BMO(\mathbb{R}^n)$, then

$$\left\| [b, S_\gamma](f) \right\|_{q_w,p,\alpha} \preccurlyeq \|f\|_{q_w,p,\alpha}.$$

Theorem [F.]

Let $0 < \gamma \leq 1$, $1 < q \leq \alpha < p \leq \infty$ and $w \in \mathcal{A}_q$. If $b \in BMO(\mathbb{R}^n)$ and $\lambda > \max\{q, 3\}$ then

$$\left\| \left[b, g^*_{\lambda,\gamma}\right](f) \right\|_{q_w,p,\alpha} \preccurlyeq \|f\|_{q_w,p,\alpha}.$$

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Let $0 < \gamma \leq 1$, $1 < q \leq \alpha < p \leq \infty$ and $w \in \mathcal{A}_q$. If $b \in BMO(\mathbb{R}^n)$ and $\lambda > \max\{q, 3\}$ then

$$\left\|\left[b, g^*_{\lambda,\gamma}\right](f)\right\|_{q_w,p,\alpha} \preccurlyeq \|f\|_{q_w,p,\alpha}\,.$$

Some references on these spaces

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THANKS