Parametric estimation for sub-fractional Ornstein-Uhlenbeck process

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Introduction

$$X_0 = 0, \;\; dX_t = \theta X_t dt + dS_t^H, \quad t \geq 0 \qquad (1.1)$$

S^H is a sub-fbm with Hurst index $H > \frac{1}{2}$ and $\theta \in (-\infty, \infty)$.

Introduction

- S^H is a BM, $\hat{\theta}$ has been studied by LIPTSER and SHIRYAEV 2001, KUTOYANTS 2004, BASAWA and SCOTT 1983, DIETZ and KUTOYANTS 2003.
- S^H is an α-stable Lévy motion in the equation (1.1), HU and LONG 2007 studied $\hat{\theta}$
- S^H FBM, $\hat{\theta}$ has been studied by LEBRETON 1998, LEBRETON 2002, PRAKASA RAO 2008, HU and NUALART 2010, BELFADI 2011.

Introduction

Our goal is to study $\hat{\theta}$, in the case $\theta > 0$, by using the least squares estimator defined by

$$\hat{\theta}_t = \frac{\int_0^t X_s dX_s}{\int_0^t X_s^2 ds}, \qquad (1.2)$$

where the integral $\int_0^t X_s dX_s$ is interpreted as a Young integral.
This least squares estimator is obtained by the least squares technique, that $\hat{\theta}_t$ minimizes

$$\theta \mapsto \int_0^t |\dot{X}_s + \theta X_s|^2 ds.$$

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Preliminaries

Definition

- BOJDECKY 2004 introduced and studied a rather special class of self-similar Gaussian processes which preserves many properties of fractional Brownian motion.
- Sub-fBm with index $H \in (0,1)$ is mean zero Gaussian process $\{S_t^H, t \geq 0\}$ with $S_0^H = 0$ and the covariance

$$C_H(t,s) = \mathbb{E}[S_t^H S_s^H] = s^{2H} + t^{2H} - \frac{1}{2}[(s+t)^{2H} + |s-t|^{2H}] \quad (2.1)$$

for all $s, t \geq 0$.

Preliminaries

Properties

- S^H is neither a semimartingale nor a Markov process unless $H = \frac{1}{2}$, so many of the powerful techniques from stochastic analysis are not available when dealing with S^H.
- The sub-fractional Brownian motion has properties analogous to those of fractional Brownian motion (self-similarity, long-range dependence, Hölder paths), and satisfies the following estimates:

$$
\begin{aligned}
&[(2 - 2^{2H-1}) \wedge 1]|t-s|^{2H} \leq \mathbb{E}|S_t^H - S_s^H|^2 \\
&\leq [(2 - 2^{2H-1}) \vee 1]|t-s|^{2H} \qquad (2.2)
\end{aligned}
$$

Preliminaries

Properties

$\mathcal{H}_{S^H}$ is the closure of the linear span $\mathcal{E}$ generated by the indicator function $\mathbf{1}_{[0,t]}, t \in [0,T]$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}_{S^H}} = C_H(t,s).$$

We know that the covariance of sub-fractional Brownian motion can be written as

$$\mathbb{E}(S_t^H S_s^H) = \int_0^t \int_0^s \phi_H(u,v)\,du\,dv = C_H(s,t) \qquad (2.3)$$

where $\phi_H(u,v) = H(2H-1)[|u-v|^{2H-2} - (u+v)^{2H-2}]$ and $\frac{1}{2} < H < 1$.

Preliminaries

Formulae (2.3) implies that

$$\langle \varphi, \psi \rangle_{\mathcal{H}_{S^H}} = \int_0^t \int_0^t \varphi_u \psi_v \phi_H(u, v) du dv \qquad (2.4)$$

for any pair step functions φ and ψ on $[0, T]$.
Isometry between the Hilbert space $\mathcal{H}_{S^H}$ and $L^2([0, T])$.

Preliminaries

The canonical Hilbert space $\mathcal{H}_{S^H}$ associate to S^H satisfies:

Lemma

$$L^2([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset \mathcal{H}_{S^H}, \tag{2.5}$$

where $H > \frac{1}{2}$.

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Preliminaries

Malliavin calculus associated with S^H.

We refer to NUALART 2006 for detailed account these notions.
The derivative operator D of a smooth cylindrical random variables $F = f(S^H(\varphi_1), ..., S^H(\varphi_n))$ is defined as the $\mathcal{H}_{S^H}-$valued random variable

$$DF = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(S^H(\varphi_1), ..., S^H(\varphi_n))\varphi_j.$$

Preliminaries

Malliavin calculus associated with S^H.

For any integer we denote by $\mathbb{D}^{k,p}$ $(k, p \geq 1)$ the Sobolev space defined as the closure of the space of smooth random variables with respect to the norm

$$\|F\|_{k,p}^{p} = \mathbb{E}(|F|^p) + \sum_{j=1}^{k} \|D^j F\|^{p}_{L^p(\Omega, \mathcal{H}_{S^H}^{\otimes j})}.$$

Preliminaries

Malliavin calculus associated with S^H.

Consider the adjoint δ of D in $L^2(\Omega, \mathcal{H}_{S^H})$. Its domain is the class of elements $u \in L^2(\Omega, \mathcal{H}_{S^H})$ such that

$$\mathbb{E}(\langle DF, u\rangle_{\mathcal{H}_{S^H}}) \leq C\|F\|_{L^2(\Omega)},$$

for any $F \in \mathbb{D}^{1,2}$, and $\delta(u)$ is the element of $L^2(\Omega)$ given by

$$\mathbb{E}(\delta(u)F) = \mathbb{E}(\langle DF, u\rangle_{\mathcal{H}_{S^H}})$$

for any $F \in \mathbb{D}^{1,2}$.

Preliminaries

Malliavin calculus associated with S^H.

We will make use the notation

$$\delta(u) = \int_0^T u_s \delta S_s^H, u \in Dom(\delta).$$

It is well-known that $\mathbb{D}^{1,2}(\mathcal{H}_{S^H})$ is included in the domain of δ.

Preliminaries

Malliavin calculus associated with S^H.

Note that $\mathbb{E}(\delta(u)) = 0$ and the variance of $\delta(u)$ is given by

$$\mathbb{E}(\delta(u)^2) = \mathbb{E}(\|u\|^2_{\mathcal{H}_{S^H}}) + \mathbb{E}(\langle Du, (Du)^* \rangle_{\mathcal{H}_{S^H} \otimes \mathcal{H}_{S^H}}), \qquad (2.6)$$

if $u \in \mathbb{D}^{1,2}(\mathcal{H}_{S^H})$, where $(Du)^*$ is the adjoint of Du in the Hilbert space $\mathcal{H}_{S^H} \otimes \mathcal{H}_{S^H}$.

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Malliavin calculus associated with S^H

We also need the commutativity relationship between D and δ

$$D\delta(u) = u + \int_0^1 Du_s \delta B_s, \qquad (2.7)$$

if $u \in \mathbb{D}^{1,2}(\mathcal{H}_B)$ and the process $\{Du_s, s \in [0,1]\}$ belong to the domain of δ.

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Preliminaries

Young integral with S^H

Let $f, g : [0, T] \to \mathbb{R}$ are Hölder continuous functions of order $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ with $\alpha + \beta > 1$.

- YOUNG 1936 proved that the Riemman-Stiltjes (so called Young integral) $\int_0^T f(s)dg(s)$ exists.
- If $\alpha = \beta \in (\frac{1}{2}, 1)$ and $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a function of class $\mathcal{C}^1$, the integrals

$$\int_0^{\cdot} \frac{\partial \phi}{\partial f}(f(u), g(u))df(u)$$

and

$$\int_0^{\cdot} \frac{\partial \phi}{\partial g}(f(u), g(u))dg(u)$$

exist in Young sense.

Preliminaries

Young integral with S^H

and the following change variables holds:

$$\begin{aligned}\phi(f(t),g(t)) &= \phi(f(0),g(0)) + \int_0^t \frac{\partial\phi}{\partial f}(f(u),g(u))df(u) \\ &+ \int_0^t \frac{\partial\phi}{\partial g}(f(u),g(u))dg(u),\end{aligned}$$

for $0 \leq t \leq T$.

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Young integral with S^H

As a consequence, if $\frac{1}{2} < H < 1$ and $(u_t, t \in [0, T])$ be a process with Hölder paths of order $\alpha > 1 - H$, the integral $\int_0^T u_s dS_s^H$ is well-defined as young integral.

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Young integral with S^H

Suppose moreover that for any $t \in [0, T]$, $u_t \in \mathbb{D}^{1,2}$, and

$$\mathbb{P}\left(\int_0^T \int_0^T |D_s u_t||t-s|^{2H-2} dsdt < \infty\right) = 1.$$

Then, by the same argument as in ALOS and NUALART 2003 we have

$$\int_0^t u_s dS_s^H = \int_0^t u_s \delta S_s^H + \int_0^t \int_0^t D_s u_r \phi_H(s,r) dsdr. \qquad (2.8)$$

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Young integral with S^H

In particular, when u is a non-random Hölder continuous function of order $\alpha > 1 - H$, we obtain

$$\int_0^t u_s dS_s^H = \int_0^t u_s \delta S_s^H. \tag{2.9}$$

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The linear equation (1.1) has the following explicit solution:

$$X_t = e^{\theta t} \int_0^t e^{-\theta s} dS_s^H, \;\; t \geq 0 \tag{3.1}$$

Let us introduce the following process

$$Y_t = \int_0^t e^{-\theta s} dS_s^H, \;\; t \geq 0.$$

By using the equation (1.1) and (3.1) we can write the least square estimator $\hat{\theta}_t$ defined in (1.2) as follows

$$\hat{\theta}_t = \theta + \frac{\int_0^t X_s dS_s^H}{\int_0^t X_s^2 ds} = \theta + \frac{\int_0^t e^{s\theta} Y_s dS_s^H}{\int_0^t e^{2\theta s} Y_s^2 ds} \tag{3.2}$$

Main resultats

Consistency of the estimator LSE

The following theorem proves the strong consistency of the LSE $\hat{\theta}_t$.

Theorem

Assume $H \in (\frac{1}{2}, 1)$, *then*

$$\hat{\theta}_t \to \theta \textit{ almost surely}, \ \textit{as } t \to \infty.$$

Main resultats

Consistency of the estimator LSE

For the proof of theorem we need the following two lemmas.

Lemma

Suppose that $H \in (\frac{1}{2}, 1)$. Then

i) *For all $\varepsilon \in (0, H)$, the process Y admits a modification with $(H-\varepsilon)$-Hölder continuous paths, still denoted Y in the sequel.*

ii) $Y_t \to Y_\infty = \int_0^\infty e^{-\theta r} dS_s^H$ *almost surely and in $L^2(\Omega)$ as $t \to \infty$.*

Main resultats

Consistency of the estimator LSE

Lemma

Suppose that $H \in (\frac{1}{2}, 1)$. Then, as $t \to \infty$

$$e^{-2\theta t}\int_0^t X_s^2 ds = e^{-2\theta t}\int_0^t e^{2\theta s} Y_s^2 ds \to \frac{Y_\infty^2}{2\theta} \quad \textit{almost surely.}$$

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Main resultats

Asymptotic distribution of the estimator LSE

Theorem

Assume $H > \frac{1}{2}$. Then, as $t \to \infty$,

$$e^{\theta t}(\hat{\theta}_t - \theta) \overset{law}{\to} \frac{2\theta\ell(H)}{\mathbb{E}[Y_\infty^2]}\mathcal{C}(1),$$

where $\mathcal{C}(1)$ the standard Cauchy distribution and $\ell(H)$ is the limiting variance of $e^{-\theta t}\int_0^t e^{\theta s} dS_s^H$.

Main resultats

Asymptotic distribution of the estimator LSE

In order to prove Theorem we need the following two lemmas.

Lemma

Assume $H > \frac{1}{2}$. Let F be any $\sigma(S^H)-$mesurable random variable such that $\mathbb{P}(F < \infty) = 1$. Then, as $t \to \infty$,

$$\left(F, e^{-\theta t}\int_0^t e^{\theta s} dS_s^H\right) \overset{law}{\to} (F, \ell(H)N)$$

where $N \sim \mathcal{N}(0,1)$ is independent of S^H and $\ell(H)$ is the limiting variance of $e^{-\theta t}\int_0^t e^{\theta s} dS_s^H$.

Asymptotic distribution of the estimator LSE

Lemma

Assume $H > \frac{1}{2}$. Then, as $t \to \infty$,

$$e^{-\frac{\theta t}{2}} \int_0^t \delta S_s^H e^{-\theta s} \int_0^s \delta S_r^H e^{\theta r} \to 0 \text{ in } L^2(\Omega) \tag{3.3}$$

and

$$e^{-\frac{\theta t}{2}} \int_0^t dse^{-\theta s} \int_0^s dre^{\theta r} \phi_H(s, r) \to 0 \tag{3.4}$$